Lesson 04: Newton's laws of motion

If you are not familiar with the basics of calculus and vectors, please read our freely available lessons on these topics, before reading this lesson.

Dynamics – an introduction

- In kinematics, we described motion (of matter) – how is a body moving?
- In dynamics, we relate the motion to its cause, that is forces. Why is the body moving in a particular way?
- Three laws of motion described by the scientist Newton in the 17^{th} century is the foundation of dynamics.
- These laws are based on experimental observations – they are not mathematically derived from something more fundamental.

What is force?

- We associate force with a push or a pull – when we push or pull a body, we say that we apply a force to the body.
- Note there is an interaction between two bodies – the body applying the force (called body A), and the body on which the force is applied (called body B).
  - For a force to act on a body, some other body must be present. We must be able to answer the question – which body is applying the force?
  - The two bodies may be in physical contact or not – so we have contact forces and long range forces.
    - Contact forces occur between bodies in contact (e.g. the force of friction or tension in a string). Contact forces, at the molecular level, are actually electrical forces.
    - Long range forces refer to gravitational and electrical forces where the bodies exerting (& hence experiencing) these forces can be separated from each other, even by empty space.

Newton's first law of motion

- First law states that in the absence of external interactions (forces), a body P remains in a state of uniform motion (i.e. **v** = constant, which may be 0 (body P at rest); and **a** = 0). This property of a body to remain in uniform motion in the absence of external forces is called **inertia**.
- Note **v** is measured with respect to a coordinate system (frame of reference).
  a) We know that $\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA}$.
  b) If body P has no external interactions and $\mathbf{v}_{PA}$ is constant, then Newton's first law is valid from A's frame of reference. But if $\mathbf{v}_{BA}$ is not constant, then $\mathbf{v}_{PB}$ is also not constant (since their sum has to be the constant $\mathbf{v}_{PA}$). This means that the first law is invalid in B's frame of reference.
  c) We call A as an **inertial frame of reference** – it is a frame of reference in which Newton's first law holds. The earth is a good approximation to an inertial frame of reference.
  d) In point b), if $\mathbf{v}_{BA}$ is constant, i.e. B is moving at a constant velocity relative to A, then $\mathbf{v}_{PB}$ is also constant. That is Newton's first law holds in B as well. So any frame of reference moving at a constant velocity with respect to an inertial frame, is also inertial.
  e) A frame of reference which is not inertial is called an **accelerated** (or **non-inertial**) **frame of reference**. e.g. B in point b) above. Note $\mathbf{v}_{BA}$ is not constant, which means B is accelerating with respect to the inertial frame A.

Newton’s second law of motion

- In an inertial frame of reference, the following is experimentally observed.
  - Magnitude of acceleration of a body P is directly proportional to the magnitude of force applied (i.e. acceleration “a” is proportional to force F).
  - Different bodies when subject to the same force have different accelerations. For two bodies A and B subjected to the same force (of any magnitude), the ratio of their accelerations is a constant. So presumably the acceleration is dependent on some property of the body that we call **mass**.
    - Since we haven’t yet defined how to measure force, we can think of the same force being exerted, when we pull a body using a specific spring stretched to a certain length. Using two similar springs side by side, stretched to the same extent is equivalent to doubling the force.
- If m_A and m_B is the mass of body A and B respectively, we define their ratio $m_A/m_B = a_B/a_A$ where a_A and a_B is the magnitude of the acceleration of A and B respectively, when acted on by the same force. So we now know, how to compare masses.
  - Acceleration “a” is inversely proportional to mass m – greater the mass, the lesser is its acceleration (i.e. it shows greater resistance to change its state of uniform motion). Therefore, mass is a measure of inertia.
- We can pick any object as representing unit mass – in the SI unit, a mass of 1kg is defined as that of a cylinder maintained by the standards body.
  - With unit mass defined, we can assign a mass to any other body by comparing accelerations as given above.
  - Tying body A and B together, the mass found based on comparing accelerations is $m_A + m_B$. Thus mass represents the “quantity of matter” in a body.

Newton's second law (continued)

- Based on the experimental observation of acceleration proportional to force, and the definition of mass; we can write a = cF/m, where c is a constant.
- We take c as 1, so that one unit of force produces one unit of acceleration, when acting on a unit mass. Thus F = ma.
  - In the SI system, the unit of force 1 N (newton) is defined as the force which produces an acceleration of 1 m/s^2, when acting on a mass of 1kg.
- It is also experimentally verified that
  - The acceleration that a force F produces, has the same direction as F. Hence we can write **F** = m**a**.
  - Forces obey the **principle of superposition**: The acceleration **a** from several forces acting on a body is the vector sum of the accelerations produced by each force acting alone.

$$\mathbf{a} = \sum_{i=1}^{n} \mathbf{a}_i = \sum_{i=1}^{n} \frac{\mathbf{F}_i}{m} \rightarrow \sum_{i=1}^{n} \mathbf{F}_i = m\mathbf{a} \quad \text{(applies only in an inertial frame of reference)}$$

This is Newton's 2nd law. The mass times the acceleration of a body is equal to the vector sum of the forces (total force) acting on the body. This confirms the vector nature of force.

It is usually applied in component form to solve problems, i.e.

$$\sum F_x = ma_x \quad \sum F_y = ma_y \quad \sum F_z = ma_z$$

Newton's third law of motion

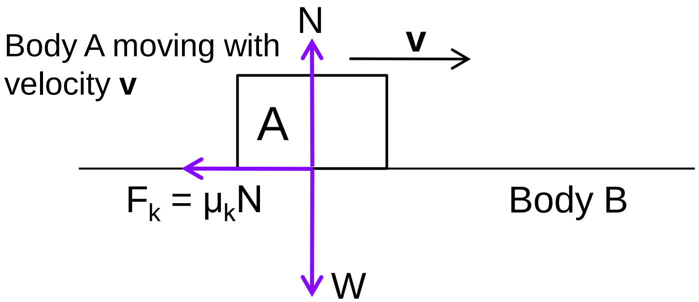
- The third law is a statement of the fact that forces arise because of interactions between bodies. It states that if body B exerts a force $\mathbf{F}_A$ on body A, then body A exerts a force $\mathbf{F}_B$ on body B such that $\mathbf{F}_B = -\mathbf{F}_A$.
  - Forces always appear in pairs. Sometimes, you call them as action and reaction, but this does *not* mean that the reaction force is a result of the action force (there is no cause – effect relationship between the forces). Either force can be called the action, and the other one is the reaction.
  - Note, the forces equated above act on different bodies. When we apply Newton's 2^{nd} law to a body, only one of the forces in an action-reaction pair (namely that which acts on the body) contributes to $\sum\mathbf{F}$.

Weight of a body

- We know from the kinematics lesson, that a body falling under the force of earth's gravity, has a constant acceleration of magnitude $g = 9.8\ m/s^2$.
- From Newton's 2^{nd} law, the gravitational force on the body has a magnitude mg, where m is the mass of the body. We call this force as the **weight** of the body, and its magnitude $W = mg$.
- When the weight of body A and body B are equal, it implies their mass is equal. The familiar equal-arm balance determines when two weights are equal.

Contact forces – normal force and friction

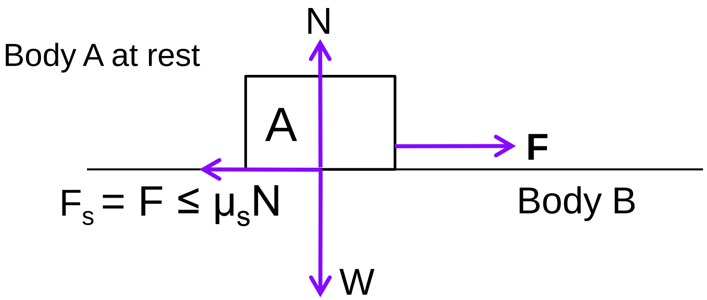
- When a body is in contact with a surface, the surface exerts a force on the body. This force consists of two components – one perpendicular to the surface called the **normal force** and another parallel (tangential) to the surface called **friction**.
- Friction arises only when a body A moves or tries to move relative to the surface of a second body B.
  - The magnitude of the frictional force F_k on body A when it is moving relative to B, is given approximately by $F_k = \mu_k N$, where N is the normal force and μ_k is called the **coefficient of kinetic friction**. μ_k is determined experimentally and depends on the nature of the surfaces in contact. F_k is in the direction opposite to the motion of body A and is called **kinetic friction**.



Normal force and friction (continued)

- When we apply a small force F to a body at rest, it doesn't move. This is because of the frictional force F_s opposing its motion. F_s is called **static friction**. As we increase the applied force F, static friction F_s also increases, keeping the body at rest. But F_s has a certain maximum value, and if F increases beyond this, the body starts to move. This maximum value of $F_s = \mu_s N$, where N is the normal force and μ_s is called the **coefficient of static friction**. μ_s is also determined experimentally and depends on the nature of the surfaces in contact.
- Note $F_s \le \mu_s N$ where the equality holds only when the body is about to move (that is F_s has attained its maximum value).
- For a given pair of surfaces, $\mu_s > \mu_k$ – hence the maximum force of static friction is greater than the kinetic friction.

- The normal force is what "balances" W, the weight of body A. The normal force is due to inter-molecular repulsions (electrical forces) between the surfaces in contact. Frictional force is also electrical in nature (at the molecular level).



Applying Newton’s laws

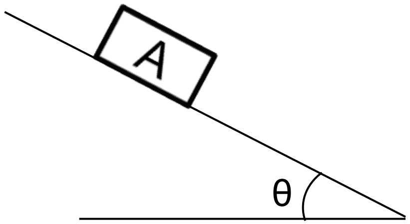
- Based on the forces applied to a body, we can use Newton’s laws to determine how the body moves.
- The problems where we apply the laws, consist of a collection of bodies that interact with each other.
  a) We take each body in turn, and draw a force diagram (also called a **free – body diagram**) showing the magnitude and direction of each force acting on it. For each force, we must be able to answer the question – which body exerts this force?
  b) In the force diagram of body A, when we show a force F exerted on it by body B; then in the force diagram of body B, we can show the force F in the opposite direction, which is exerted on it by body A (Newton’s third law).
  c) We setup an inertial frame of reference and for each body, we write Newton’s 2nd law in component form. e.g. using the force diagram for body A, we find what is the x component of each force, based on the indicated magnitude and direction. We sum up the x components of all the forces on body A, and equate it to Ma_x, where M is the mass of body A, and a_x is the x component of its acceleration.
  d) The bodies may be constrained in some way as implied by the problem statement – write down the kinematical equations for this. e.g. the magnitude of acceleration for two bodies connected by an inextensible taut string is the same.
  e) Keep track of the known and unknown variables. Newton’s 2nd law along with the constraint equations should provide enough relations to find the unknowns.

Example 1

Body A of mass m rests on a plank. The coefficient of static friction between the plank and the body is μ_s. What is the maximum angle θ that the plank can make with the horizontal without the body slipping?

Solution: The force diagram of body A is shown along with the positive x and y axis.

- The three forces acting on body A are its weight (mg), normal force N and the static friction F_s.
- mg, N and F_s are the magnitude of these forces in the direction indicated.
- At the maximum angle, the static friction force is at its maximum, i.e. $F_s = \mu_s N$.
- a_x and $a_y = 0$ since the body is at rest.

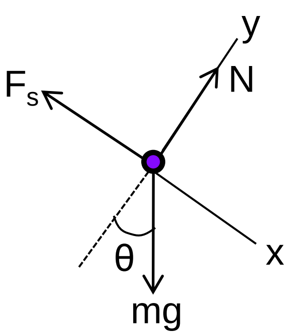


$$1)\ mg\sin\theta - \mu_s N = 0 \quad \text{(x components should add up to 0)}$$

$$2)\ N - mg\cos\theta = 0 \quad \text{(y components should add up to 0)}$$

$$\therefore mg\sin\theta = \mu_s N = \mu_s mg\cos\theta$$

$$\text{So } \tan\theta = \mu_s \text{ or } \theta = \tan^{-1}\mu_s$$



Notes:

- mgsinθ is the component of weight along the x axis.
- $-F_s = -\mu_s N$ is the component of friction along the x axis.
- $\sum F_x = 0$ gives the first equation above.

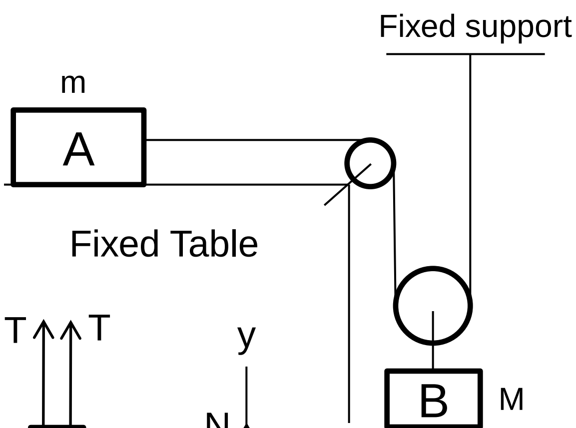
- N is the component of normal force along the y axis.
- -mgcos θ is the component of weight along the y axis.
- $\sum F_y = 0$ gives the second equation above.

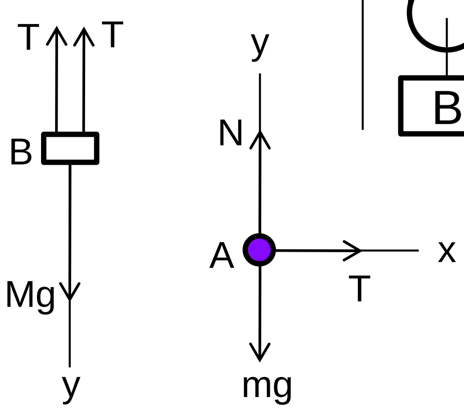
Example 2

In the diagram shown, there is no friction. The pulleys are light and frictionless. The string is light and inextensible. Block A and Block B have mass m and M respectively. Find the acceleration of each block in terms of m, M and g.

Solution: The force diagrams for body A and body B are shown. T is the uniform tension in the string. We choose a separate coordinate axis (fixed to the table) for each body, which is okay.

Note when A moves a distance x to the right, B will move down by x/2. With our choice of axes, this means that a_x for A is twice a_y for B. Let us denote acceleration a_y of B as "a".





For body B

$Mg - 2T = Ma$ (Newton's 2nd law for y components)

For body A

$T = m(2a)$ (Newton's 2nd law for x component)

$$\therefore Mg - 4ma = Ma \rightarrow a = \frac{Mg}{4m + M} \quad \text{(the acceleration of mass M)}$$

The acceleration of mass m is 2a.

Note: A taut string pulls the bodies to which it is connected, and the magnitude of this force is called **tension**. It is another example of a contact force.
The tension at the two ends of a string is the same, when the string is light (its mass can be neglected).

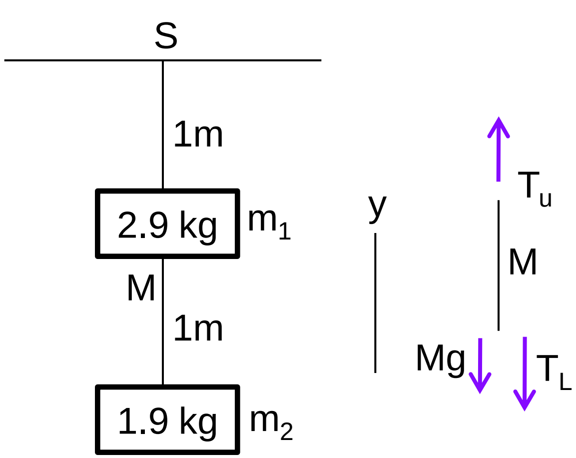
Example 3

[IIT 1989]: Two blocks of masses 2.9 kg and 1.9 kg are suspended from a rigid support S by two inextensible wires each of length 1 meter. The upper wire has negligible mass and lower wire has a uniform mass of 0.2 kg/m. The whole system of blocks and support have an upward acceleration of 0.2 m/s^2. Find the tension at the mid point of

a) the lower wire.   b) the upper wire.

Solution: Take the y axis to be vertically up (say fixed to the ground).

- Since the upper wire has negligible mass, the tension is uniform in it.
- Tension isn't uniform in the lower wire, since it has mass. To see why, consider its force diagram. If T_u is the tension at its upper end, then it pulls down on m_1 with a force T_u. Therefore, m_1 pulls the wire up with a force T_u (3^{rd} law). Similarly, if T_L is the tension at the lower end, m_2 pulls the wire down with a force T_L. Therefore, the 2^{nd} law for the wire in the y direction gives: $T_u – T_L – Mg = Ma$ (M is the mass of wire). For the wire to accelerate up (as given in the problem), T_u has to be greater than $(T_L + Mg)$ by the amount Ma. Only when M is 0, $T_u = T_L$.



- Since we need to find the tension at the mid-point of the lower wire & since the whole system is moving up with the same acceleration, we can take one body as m_1 + top half of the lower wire (mass M/2). The other body is m_2 + the lower half of the lower wire.
- The force diagrams and the equations of motion are shown on the next page.

Example 3 (continued)

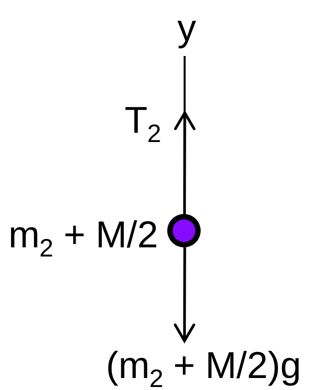
$$T_2 - (m_2 + \frac{M}{2})g = (m_2 + \frac{M}{2})a \quad \text{(2nd law in y direction for composite body } m_2 + \frac{M}{2})$$

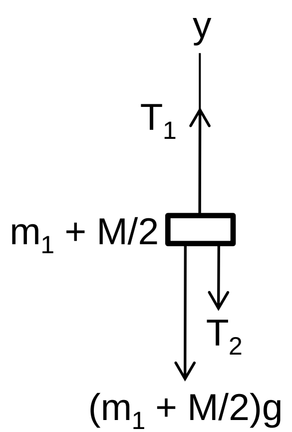
$$\therefore T_2 = (m_2 + \frac{M}{2})(g + a) = (1.9 + 0.1)(9.8 + 0.2) = 20N \text{ (tension at midpoint of lower wire).}$$

Note half of the lower wire has mass 0.1 kg, since the whole wire is 1 m long and the mass per unit length = 0.2 kg/m.

$$T_1 - T_2 - (m_1 + \frac{M}{2})g = (m_1 + \frac{M}{2})a \quad \text{(2nd law in y direction for composite body } m_1 + \frac{M}{2})$$

$$\therefore T_1 = T_2 + (m_1 + \frac{M}{2})(g + a) = 20 + (2.9 + 0.1)(9.8 + 0.2) = 50N \text{ (tension throughout the upper wire)}$$





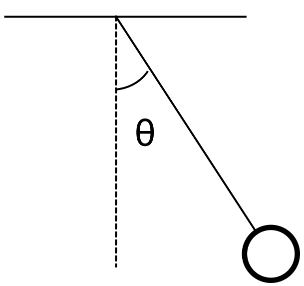
Example 4

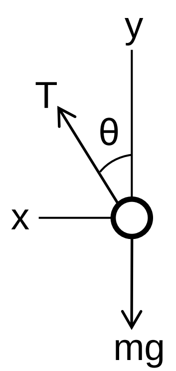
[IIT 1992]: A car is moving in a circular horizontal track of radius 10 m with a constant speed of 10 m/s. A plumb bob is suspended from the ceiling of the car by a light rigid rod of length 1 m. The angle made by the rod with the vertical is (take g = 10 m/s^2).

a) zero b) 30° c) 45° d) 60°

Solution: Using an inertial frame on the ground, the bob has a centripetal acceleration of v^2/r. The force for this comes from the horizontal component of the tension in the rod as shown in the force diagram.

The force responsible for the centripetal acceleration is called the **centripetal force**.





$$T\cos\theta - mg = 0 \quad \text{(2nd law for y components)}$$

$$T\sin\theta = \frac{mv^2}{r} \quad \text{(2nd law for x components)}$$

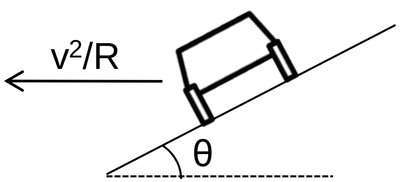
$$\therefore \tan\theta = \frac{v^2}{rg} = \frac{10^2}{10\times 10} = 1 \rightarrow \theta = 45^\circ\text{. So the answer is c)}$$

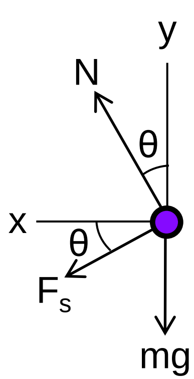
Example 5 (Banked curves)

When a vehicle (say a car) moves around a circular path, some force must act on the car towards the centre, to produce centripetal acceleration. This force is static friction, since there is no motion in the radial direction; and the car moves tangential to the circle.

Sometimes the road can be banked, that is the road slopes down from the outer radius to the inner radius, so that the horizontal component of the normal force also contributes to the centripetal force. What is the maximum speed the car can have, to maintain circular motion (and not skid)?

Solution: R is the radius of the path. θ is the banking angle and v is the speed of the car. Let us represent the frictional force F_s in the direction shown by μN, where $-\mu_s \le \mu \le \mu_s$, where μ_s is the coefficient of static friction. This may seem strange, but it only means that the frictional force can be downwards as shown, or act upwards ($\mu < 0$), and its maximum magnitude is $\mu_s N$.





1) $N\cos\theta - \mu N\sin\theta - mg = 0$ (2nd law for y components)

2) $N\sin\theta + \mu N\cos\theta = mv^2/R$ (2nd law for x components)

From 1) we have $N = mg/(\cos\theta - \mu\sin\theta)$. Substituting for N in 2)

$$v^2 = \frac{Rg(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta} \quad (3) \rightarrow v_{\max}^2 = \frac{Rg(\mu_s + \tan\theta)}{1 - \mu_s\tan\theta} \quad (\text{maximum value when } \mu = \mu_s)$$

When $\theta = 0$ (no banking) $v_{\max} = \sqrt{\mu_s Rg}$

Setting $\mu = 0$ in 3) gives the speed $v_0 = \sqrt{Rg\tan\theta}$ at which the normal force alone provides the required centripetal force. If $v < v_0$, then $\mu < 0$ in 3), and the frictional force acts up the slope.

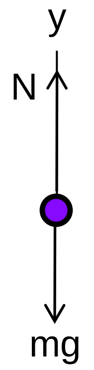
Example 6 (Apparent weight)

A person of mass m stands on a bathroom (weighing) scale in an accelerating elevator. What is the scale reading?

Solution: The person is acted on by two forces – his weight (mg) and the normal force N from the scale. By Newton's 3rd law, he applies an equal force N on the scale, and this is what the scale reads. From Newton's 2nd law

$N - mg = ma_y$ which implies $N = m(g + a_y)$

- If the lift (and hence the person) is accelerating up (a_y positive); N (and hence the scale reading) is more than mg.
- If the lift is accelerating down (a_y negative), the scale shows a lower weight than mg. When $a_y = -g$ (free fall), N = 0 (apparently weightless).
- The scale reading is called the **apparent weight**.



Newton's law in a non-inertial frame

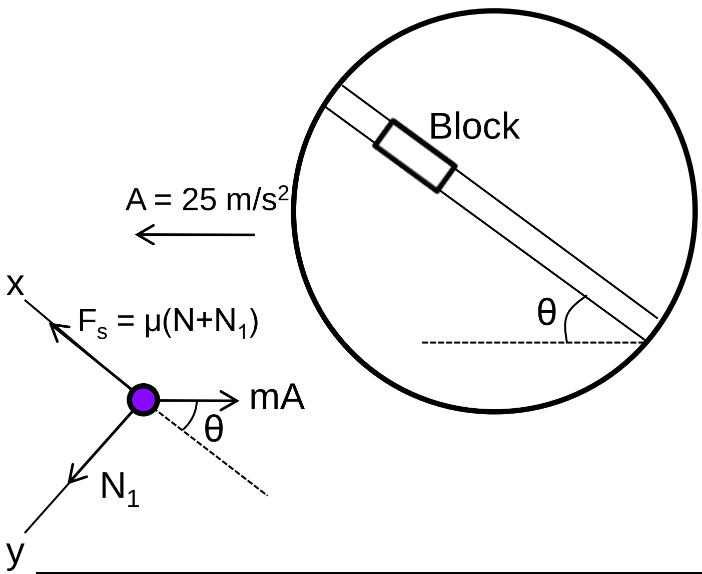
- We have seen that the relative velocity of a point P as per two observers A and B is related by $\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA}$.
- Differentiating this with respect to time, we have $\mathbf{a}_{PA} = \mathbf{a}_{PB} + \mathbf{a}_{BA}$. Let A be an inertial frame, and $\mathbf{a}_{BA}$ is not zero (that is B is accelerating relative to A, and hence is a non-inertial frame).
- So $\mathbf{a}_{PB} = \mathbf{a}_{PA} – \mathbf{a}_{BA}$. Multiply by mass M of point P, and note that $M\mathbf{a}_{PA}$ is equal to **F**, the true force acting on P, since A is an inertial frame.
- We get $M\mathbf{a}_{PB} = \mathbf{F} – M\mathbf{a}_{BA}$. This means that M**a** in B's frame is equal to **F** (the actual force) minus $M\mathbf{a}_{BA}$ where $\mathbf{a}_{BA}$ is the acceleration of B relative to an inertial frame. Thus while applying Newton's 2nd law in a non-inertial frame B, we need to add $–M\mathbf{a}_{BA}$ to the force side of the equation. This quantity –$M\mathbf{a}_{BA}$ is called a **pseudo** or **fictitious** force, because it is not a true force arising from interaction between bodies.
  - In $\mathbf{a}_{BA}$, the acceleration of B can be relative to any inertial frame, because if A and A' are two inertial frames, we have seen that they have a constant velocity relative to each other. Differentiating $\mathbf{v}_{BA} = \mathbf{v}_{BA'} + \mathbf{v}_{A'A}$ with respect to t, we get $\mathbf{a}_{BA} = \mathbf{a}_{BA'}$ since $\mathbf{v}_{A'A}$ is constant (implies acceleration of B is the same in all inertial frames).

Example 7

[IIT 2006]: A circular disc with a groove along its diameter is placed on a horizontal surface. A block of mass 1 kg is placed in the groove. The coefficient of friction between the block and the groove surfaces is μ = 2/5. The disc moves with an acceleration of 25 m/s^2 as shown. What is the acceleration of the block with respect to the disc? Take g = 10 m/s^2 and sinθ = 3/5 and cosθ = 4/5.

Solution:

- The disc is a non-inertial frame with acceleration A = 25 m/s^2, hence we introduce a pseudo force -mA in the force diagram for the block, where m is the mass of the block.
- In the direction perpendicular to the page, the weight mg is balanced by the normal force N.
- Additionally there is a normal force N_1 from the sides of the groove.
- The xy axes is parallel and perpendicular to the groove as shown.



$$\mu(mg + N_1) - mA\cos\theta = ma_x \quad \text{(2nd law for x components)}$$

$$N_1 - mA\sin\theta = 0 \quad \text{(2nd law for y components)}$$

$$\therefore a_x = \mu(g + A\sin\theta) - A\cos\theta = \frac{2}{5}(10 + 25\times\frac{3}{5}) - 25\times\frac{4}{5}$$

$$= 10 - 20 = -10\,m/s^2 \quad \text{(negative sign implies its directed along -x axis).}$$