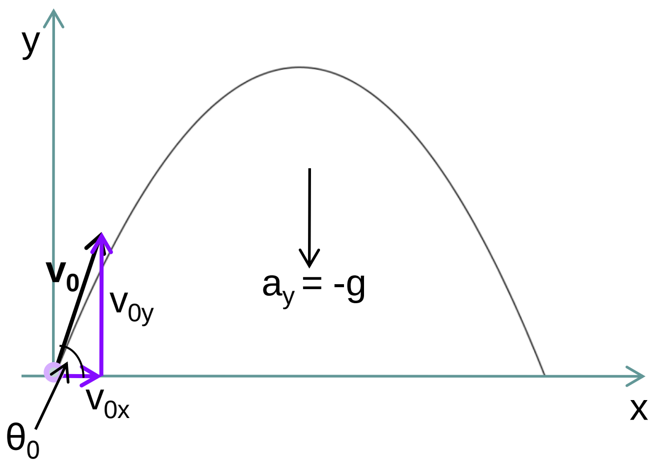
Lesson 03: Kinematics

Translational motion (Part 2)

If you are not familiar with the basics of calculus and vectors, please read our freely available lessons on these topics, before reading this lesson.

Projectile motion

- Projectile motion: Refers to the motion of a body that is given an initial velocity $\mathbf{v_0}$ (not necessarily vertical), and then moves under the action of the gravitational force. The body is called a **projectile**.
- $\mathbf{v_0}$ can be split into two components – one vertical (parallel to the gravitational force) and one horizontal (perpendicular to the gravitational force, and hence parallel to the ground).
  - x axis is defined in the horizontal direction, and y axis is defined in the vertical direction.
  - v_0 is the magnitude of the initial velocity and it makes an angle θ_0 with the x axis. Hence the y-component of the initial velocity is $v_{0y} = v_0\sin\theta_0$, and the x-component is $v_{0x} = v_0\cos\theta_0$.



- The motion of the projectile is in the vertical xy plane (as defined by its initial velocity), since gravity can only change the y component of velocity.
  - z coordinate is always 0.
  - x component of velocity is constant since there is no acceleration along the x axis.
  - y component of acceleration is constant, and is –g.
- The origin coincides with the starting point of the motion.

Projectile motion (continued)

Since each component can be considered independently

$v_x = v_{0x} = v_0 \cos\theta_0$ (no acceleration is x direction)

$v_y = v_{0y} - gt = v_0 \sin\theta_0 - gt$ (uniform acceleration of -g in y direction)

$x = v_{0x}t = (v_0 \cos\theta_0)t$

$$y = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

When the projectile reaches the maximum height, v_y is 0.

$\therefore 0 = v_0 \sin\theta_0 - gt \rightarrow t = \dfrac{v_0 \sin\theta_0}{g}$ (the time taken to reach the maximum height)

A ball thrown vertically up reaches the maximum height at t = $\dfrac{v_0}{g}$. The above expression is the same with v_0 replaced by the vertical component $v_0 \sin\theta_0$.

The maximum height H reached by the projectile can be found by substituting t = $\dfrac{v_0 \sin\theta_0}{g}$ in the expression for y.

$$H = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2 = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Projectile motion (continued)

Setting y = 0 in the expression for y, we get the time t of flight of the projectile

$$0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 = t(v_0 \sin\theta_0 - \frac{gt}{2}) \;\rightarrow\; \text{t} = \frac{2v_0 \sin\theta_0}{g}.$$

The range R of the projectile (which is the horizontal distance covered during the flight period) is

$$R = v_0 \cos\theta_0 \left(\frac{2v_0 \sin\theta_0}{g}\right) = \frac{v_0^2 \sin 2\theta_0}{g}$$ (where we have used the identity $\sin 2\theta = 2\sin\theta\cos\theta$)

R is maximum (for a given v_0) when $\sin 2\theta_0 = 1 \rightarrow \theta_0 = 45°$.

Using $x = (v_0 \cos\theta_0)t$ to eliminate t from the equation for y, we get the equation of the curve traced out by the projectile.

$$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)\left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2 = x\tan\theta_0 - \frac{gx^2}{2v_0^2 \cos^2\theta_0}$$

This is the equation of a parabola. Note it is of the form $y = ax - bx^2 = -b(x - \frac{a}{2b})^2 + \frac{a^2}{4b}$

As expected, the parabola opens down and its vertex is at $(\frac{a}{2b}, \frac{a^2}{4b}) = (\frac{v_0^2 \sin 2\theta_0}{2g}, \frac{v_0^2 \sin^2\theta_0}{2g})$,

which is the xy coordinate of the topmost point in the path.

Example 1

A stone is launched horizontally from the top of Cliff 1 with speed v_0. Assume the face of Cliff 1 and Cliff 2 is vertical.

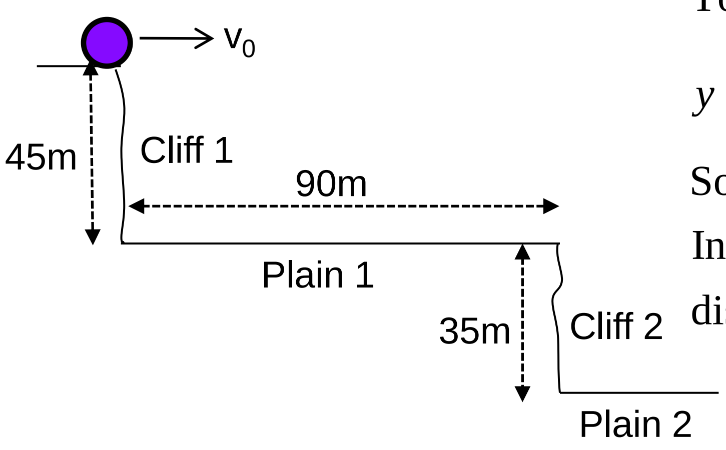
a) What is the minimum speed the stone should have, to land directly on Plain 2?

b) How far from the base of Cliff 2 will the stone land on Plain 2?

Take the value of g as 10 m/s^2.

Solution: For the projectile motion (of the stone), take the origin of the coordinate system at top of Cliff 1 (y axis vertically up, and x axis to the right). Then the coordinates of the top of Cliff 2 that it must clear is P(90m, -45m). When the stone falls 45m, its x coordinate must be $\geq$ 90m.

$\therefore y = -\frac{gt^2}{2} = -45 \rightarrow t = 3s$. For this t, $x = v_0 t \geq 90 \rightarrow v_0 \geq 30\, m/s$ (minimum speed required).



Total time taken to reach the bottom of Cliff 2 (-80m) is given by

$$y = -\frac{gt^2}{2} = -80 \rightarrow t = 4s$$

So after reaching top of Cliff 2, the stone travels an extra 4 - 3 = 1s. In this time, the horizontal distance travelled = $v_0 t = 30m$. This is the distance at which the stone lands from the base of Cliff 2.

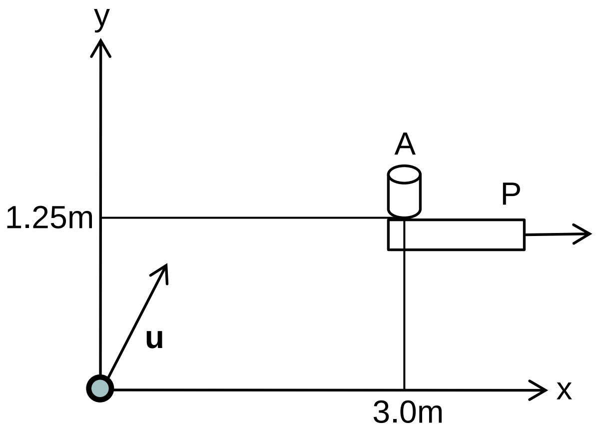
Example 2

[From IIT 2000]: An object A is kept fixed at the point x = 3m and y = 1.25m on a plank P raised above the ground. At t = 0, the plank starts moving along the +x direction with the acceleration 1.5 m/s^2. At the same instant, a stone is projected from the origin with a velocity **u** as shown in the figure. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45^o to the horizontal. All the motions are in the x-y plane. Find **u** and the time after which the stone hits the object. Take g = 10 m/s^2.

Solution: Let t be the time after which the stone hits object A, and let u_x and u_y be the x and y components of the initial velocity **u**.

When the stone hits the object after time t, the (x, y) coordinates of both stone and object are the same.

The angle θ that the velocity of the projectile makes with the horizontal is given by tanθ = v_y/v_x. This angle is 45^o when the projectile is moving down (and v_y is negative). Therefore, tan45^o = 1 = - v_y/v_x. Note there are 3 unknowns for which we can now write 3 equations as given below.



$$1)\ x = u_x t = 3 + \frac{1}{2} \times 1.5 \times t^2 \text{ (x coordinate of stone and object A are the same)}$$

$$2)\ y = u_y t - \frac{1}{2} g t^2 = 1.25 \quad \text{(y coordinate of stone and object A are the same)}$$

$$3)\ -(u_y - gt) = u_x \quad \left(\text{since } \frac{-v_y}{v_x} = 1 \text{ for the stone at the time of collision}\right)$$

Example 2 (continued)

Substitute for u_x in Eq 1 using Eq 3. Also, substitute g = 10 m/s^2. We then have

$$1a)\ -u_y t + 10t^2 = 3 + \frac{3}{4}t^2 \rightarrow -u_y t + \frac{37}{4}t^2 = 3$$

Eq 2 can be written as $u_y t - 5t^2 = \frac{5}{4}$ Adding this to 1a), we have

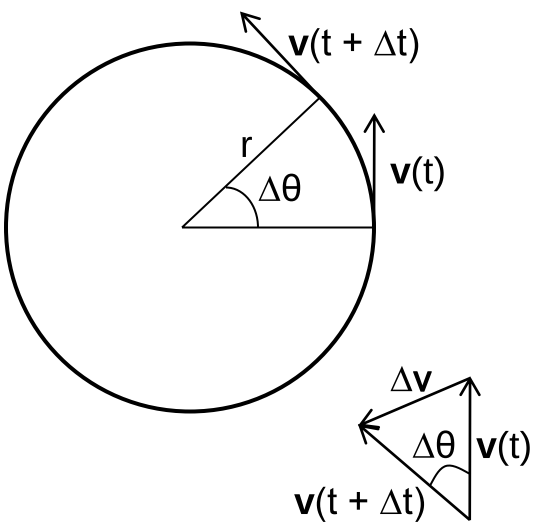
$$\frac{17}{4}t^2 = \frac{17}{4} \rightarrow t = 1s$$

Therefore, Eq 2 above gives $u_y = \frac{25}{4}$

Eq 3 give $u_x = gt - u_y = \frac{15}{4} \rightarrow \mathbf{u} = \frac{15}{4}\mathbf{i} + \frac{25}{4}\mathbf{j}\ m/s$

Uniform circular motion

- In uniform circular motion, a particle moves in a circle of radius r with constant speed v.
- In time Δt, the particle will move a distance $v\Delta t = r\Delta\theta$ (where $\Delta\theta$ is in radians).
  - $\Delta\theta/\Delta t$ is the angle turned in unit time and is called the **angular speed**, denoted by ω. Its unit is rad/s. Note $\omega = v/r$ which is a constant for uniform circular motion.
  - To go a full circle, the angle must change by 2π (equivalently, the particle must travel a distance of $2\pi r$), therefore the time T required for this $= 2\pi/\omega = 2\pi r/v$. T is called the **period** of the motion.



- The particle velocity **v** is always tangential. Though magnitude is constant, the direction changes. Hence, there is an acceleration.

From the triangle, for small $\Delta\theta$ (and hence small Δt),

$|\Delta\mathbf{v}| = v\Delta\theta$. Using the equation $v\Delta t = r\Delta\theta$, we substitute for $\Delta\theta$, to get

$$|\Delta\mathbf{v}| = \frac{v^2}{r}\Delta t \rightarrow |\mathbf{a}| = \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{v^2}{r}$$

Also, when $\Delta\theta$ approaches 0, $\Delta\mathbf{v}$ and hence **a** becomes radial, and points to the centre of the circle. It is called centripetal acceleration.

Example 3

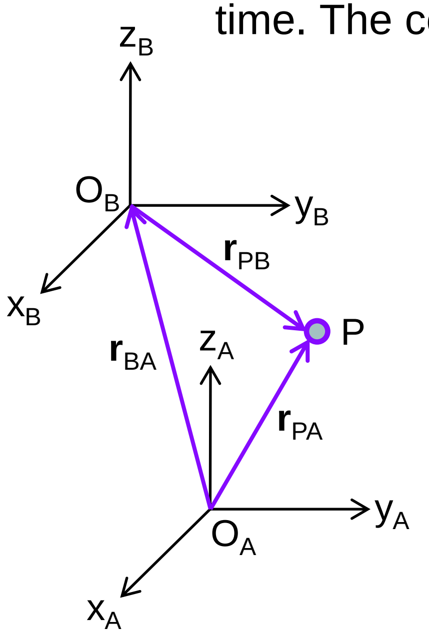
The earth has a radius of 6380 km and turns once on its axis in 24 h. What is the radial acceleration of an object at the earth's equator in m/s^2?

Solution: The radial acceleration is the centripetal acceleration $a = \frac{v^2}{r}$

$$v = \frac{2\pi r}{T} \rightarrow a = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 \times 6380 km \times 1000\, m/km}{(24h \times 3600\, s/h)^2} = \frac{4\pi^2 \times 638}{(24 \times 36)^2} m/s^2 = 0.034\, m/s^2$$

Relative velocity

- Translational motion is defined with respect to a coordinate system. When we change the coordinate system, how is the observed motion affected?
  - e.g. Consider a person P moving in a train. The translational motion of P is observed by a person A standing on the ground (outside the train) and another person B sitting in the train. How are the two observations related?
  - A and B setup their coordinate system, the axes of which are parallel, so that the corresponding unit vectors (**i**, **j** and **k**) in the two systems are the same. They use the same unit of length (say meter) on their axes. To study the motion of P, they use their coordinate axes to measure position, along with a clock to measure time. The coordinate axes with the clock is called a **frame of reference**.



- Position vector $\mathbf{r}_{PA}$ is the position of P in A's coordinate system (frame of reference) – also called the position of P relative to A.
- Position vector $\mathbf{r}_{PB}$ is the position of P relative to B.
- Position vector $\mathbf{r}_{BA}$ is the position of B relative to A. Position vector $\mathbf{r}_{AB}$ (not shown) is the position of A relative to B.

$$\mathbf{r}_{AB} = -\mathbf{r}_{BA}$$

Also from the figure, $\mathbf{r}_{PA} = \mathbf{r}_{PB} + \mathbf{r}_{BA}$

Differentiating with respect to time, we have

$\mathbf{v}_{AB} = -\mathbf{v}_{BA}$ and $\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA}$

where for example $\mathbf{v}_{PA}$ is the velocity of P in A's coordinate system.

Relative velocity (continued)

- $\mathbf{v}_{PA}$ which is the velocity of P in A's frame of reference, is also called the velocity of P relative to A (hence the term relative velocity).
- We observe the following:
  - Velocity of A relative to B ($\mathbf{v}_{AB}$) is the negative of the velocity of B relative to A ($\mathbf{v}_{BA}$).
    - e.g. if B is moving at 5 m/s in the positive x direction with respect to A, then from B's point of view, A is moving at 5 m/s in the negative x direction.
  - Velocity of P relative to A ($\mathbf{v}_{PA}$) is the sum of the velocity of P relative to B ($\mathbf{v}_{PB}$) and the velocity of B relative to A ($\mathbf{v}_{BA}$). Note the order of subscripts to remember this equation.
    - Subscript order can be readily used to extend the equation like $\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BC} + \mathbf{v}_{CA}$ where for example $\mathbf{v}_{BC}$ is the velocity of B relative to C.

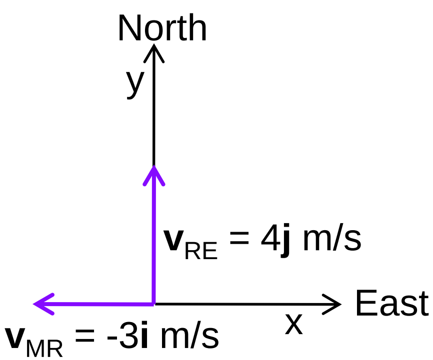
Example 4

A river flows north with a speed of 4 m/s. A man steers a boat across the river, and his velocity relative to the water is 3 m/s towards the west. The river is 600 m wide.

a) What is the velocity of the man relative to the earth?

b) In how much time, does he cross the river?

c) How far north from his starting point will he be, when he reaches the opposite bank?



Denoting man as M, earth as E and river as R; and choosing the coordinate axes as shown, we see that $\mathbf{v}_{RE} = 4\mathbf{j}\, m/s$ and $\mathbf{v}_{MR} = -3\mathbf{i}\, m/s$

a) Velocity of man relative to earth $\mathbf{v}_{ME} = \mathbf{v}_{MR} + \mathbf{v}_{RE} = -3\mathbf{i} + 4\mathbf{j}\, m/s$

b) Time T to cross the river = $\dfrac{x}{v_x} = \dfrac{600m}{3\, m/s} = 200s$

c) Distance north that he travels = $v_y T = 4\, m/s \times 200s = 800m$

Example 5

A man is walking on a straight horizontal road at 4 km/hr. Suddenly it starts to rain, and he finds that the raindrops fall on him vertically. He starts to run at 12 km/hr, and he finds that the drops fall at an angle of 45^o to the vertical. Find the speed at which the rain falls on the road.

Solution: Let us denote the man, rain and earth by M, R and E respectively. Take the x-axis along the direction in which the man is walking (or running), and the y-axis vertically down. The relative velocities when the man is walking are in Fig 1, and when he is running are in Fig 2. For example, $\mathbf{u}_{RM}$ is the velocity of rain relative to man, when he is walking. Note $\mathbf{v}_{RE}$ doesn't change whether the man walks or runs. So we have the following relationships.

$$\mathbf{v}_{RE} = \mathbf{u}_{ME} + \mathbf{u}_{RM} = 4\mathbf{i} + y\mathbf{j} \quad \text{(y has to be determined)}$$

$$\mathbf{v}_{RE} = 4\mathbf{i} + y\mathbf{j} = \mathbf{v}_{ME} + \mathbf{v}_{RM} = 12\mathbf{i} + \mathbf{v}_{RM} \rightarrow \mathbf{v}_{RM} = -8\mathbf{i} + y\mathbf{j}$$

$$\text{But } \tan 45^\circ = 1 = 8/y \rightarrow y = 8 \text{ km/hr}$$

$$\text{So } \mathbf{v}_{RE} = 4\mathbf{i} + 8\mathbf{j} \rightarrow |\mathbf{v}_{RE}| = 4\sqrt{5} \text{ km/hr.}$$

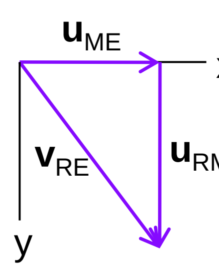


Fig 1

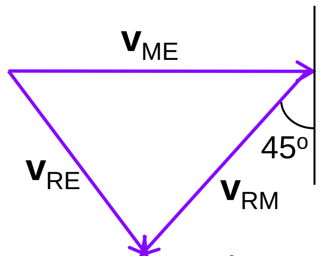


Fig 2

Projectile on an inclined plane*

We now consider projectile motion when the ground makes an angle α to the horizontal. There are two possibilities:

Fig 1: Ground slopes up in the direction of projectile motion from point O to point P.

Fig 2: Ground slopes down in the direction of projectile motion.

The length OP is called the range R of the projectile. Let us consider the scenario shown in Fig 1 first.

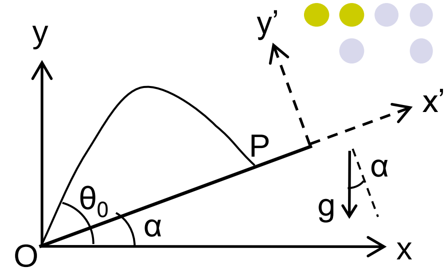


Fig 1: Projectile going up a plane

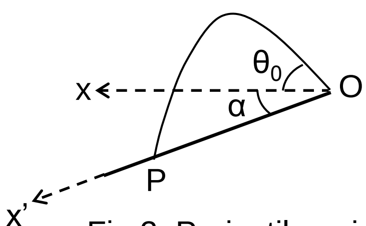


Fig 2: Projectile going down a plane

$$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \quad (1) \qquad \text{and} \qquad x = (v_0 \cos\theta_0)t \quad (2)$$

At point P, we have $y = x\tan\alpha$ (3). Using 3) and 2) in 1), we have

$$(v_0 \cos\theta_0 \tan\alpha)t = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

$$\rightarrow \text{Time of flight } t = \frac{2v_0(\sin\theta_0 - \cos\theta_0\tan\alpha)}{g} = \frac{2v_0\sin(\theta_0-\alpha)}{g\cos\alpha} \quad (4)$$

Note $v_0\sin(\theta_0 - \alpha)$ and $-g\cos\alpha$ is the component of velocity and acceleration respectively that is perpendicular to the (sloping) ground (y' components).

We can get the same result for time of flight by resolving velocity and acceleration into components along the x'y' axes, where x' is along the sloping ground, and y' is normal to it.

**This material is secondary in nature, and can be omitted on a first reading.*

Projectile on an inclined plane*(continued)

$$v_{x'} = v_0 \cos(\theta_0 - \alpha) \text{ and } v_{y'} = v_0 \sin(\theta_0 - \alpha)$$

$$a_{x'} = -g \sin\alpha \quad \text{and} \quad a_{y'} = -g\cos\alpha$$

$$y' = v_0 \sin(\theta_0 - \alpha)t - \frac{1}{2}(g\cos\alpha)t^2. \text{ At point P, } y' = 0, \text{ so time of flight } T = \frac{2v_0 \sin(\theta_0 - \alpha)}{g\cos\alpha} \text{ (as before)}$$

$$\text{Range } R = \frac{x(T)}{\cos\alpha} = \frac{v_0 \cos\theta_0 \times 2v_0 \sin(\theta_0 - \alpha)}{g\cos^2\alpha} = \frac{v_0^2\left[\sin(2\theta_0 - \alpha) - \sin\alpha\right]}{g\cos^2\alpha}$$

$$\text{Maximum R implies } \sin(2\theta_0 - \alpha) = 1. \text{ So } R_{\max} = \frac{v_0^2(1-\sin\alpha)}{g(1-\sin^2\alpha)} = \frac{v_0^2}{g(1+\sin\alpha)}$$

We can also compute R by finding x' for T, as shown below.

$$R = x'(T) = v_0\cos(\theta_0 - \alpha)T - \frac{1}{2}(g\sin\alpha)T^2 = \frac{2v_0\sin(\theta_0 - \alpha)}{g\cos\alpha}\left[v_0\cos(\theta_0 - \alpha) - \frac{1}{2}(g\sin\alpha)\frac{2v_0\sin(\theta_0 - \alpha)}{g\cos\alpha}\right]$$

$$= \frac{2v_0^2\sin(\theta_0 - \alpha)\cos\theta_0}{g\cos^2\alpha} = \frac{v_0^2\left[\sin(2\theta_0 - \alpha) - \sin\alpha\right]}{g\cos^2\alpha} \text{ (as before)}$$

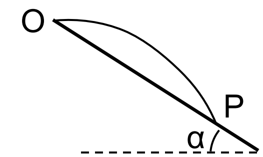
Similarly for a projectile going down a plane (Fig 2), the time of flight and maximum range is given by:

$$T = \frac{2v_0\sin(\theta_0 + \alpha)}{g\cos\alpha} \text{ and } R_{\max} = \frac{v_0^2}{g(1-\sin\alpha)}$$

Example 6*

A particle is projected horizontally with speed u, from the top of an inclined plane that makes an angle α with the horizontal. How far from the point of projection does the particle strike the plane?

Solution: We need to find the range R = OP.



Time of flight $T = \dfrac{2v_0 \sin(\theta_0 + \alpha)}{g \cos\alpha}$. Here $v_0 = u$ and $\theta_0 = 0 \rightarrow T = \dfrac{2u \tan\alpha}{g}$

Range $R = \dfrac{x(T)}{\cos\alpha} = \dfrac{uT}{\cos\alpha} = \dfrac{2u^2 \tan\alpha}{g \cos\alpha}$