Lesson 02: Kinematics

Translational motion (Part 1)

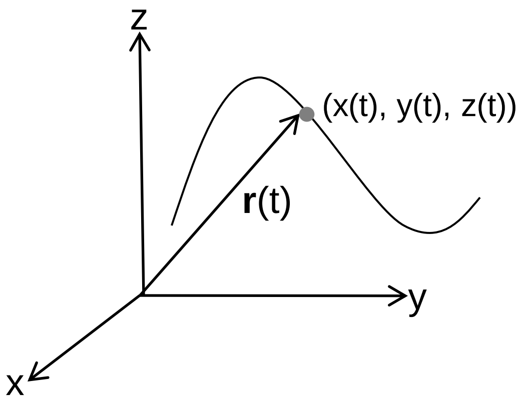
If you are not familiar with the basics of calculus and vectors, please read our freely available lessons on these topics, before reading this lesson.

Kinematics - translational motion

- Mechanics: The study of relationships between force, matter and motion.
  - Kinematics: Branch of mechanics that describes motion of a body (matter).
  - Dynamics: Branch of mechanics that relates motion (of matter) to its cause (forces).
- Translational motion: Refers to the motion of a body from one point in space to another.
  - Rotational motion: Refers to rotation about an axis (e.g. a spinning ball) as opposed to moving from one point in space to another.
  - A body is considered as a particle (a point) to describe its translational motion.
    - e.g. for a rigid body, that is a body whose shape doesn't change, and which is not rotating; any point on it can be used to describe its motion in space.
- Newtonian mechanics: Mechanics as defined by the scientist Newton. It applies to motion of bodies that we see in everyday life.
  - Not applicable at speeds close to the speed of light, and at the atomic level.

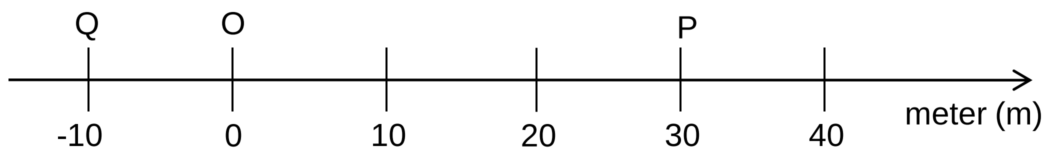
Describing translational motion

- To describe translational motion in space, we setup a coordinate system with suitable units on its axes (e.g. meter).
- The position of a particle (whose translational motion is being studied) is defined by its coordinates (x, y, z) in this coordinate system.
  - Equivalently, we can define it using its position vector **r**.
- Translational motion is described by stating how **r**, and hence x, y and z depend on time t. Knowing **r**(t) = x(t)**i** + y(t)**j** + z(t)**k** allows us to state the following about the motion of the particle:
  - What is the path (curve) along which the particle is moving?
  - How fast, and in what direction is the particle moving?
  - Is the particle slowing down or speeding up? How is the direction of motion changing?



Motion along a straight line – position, displacement and distance

- We orient one of the axis (usually x axis) along the straight line (of motion).
- The x coordinate measures the position of the particle.
  - y and z coordinates are always 0.
  - x is a function of time t, hence we write it as x(t).
- For time $t_1 < t_2$, $\Delta x = x(t_2) - x(t_1)$ is called the **displacement** of the particle in the time interval $\Delta t = t_2 - t_1$. It is the change in the position of the particle.
  - Δx (the displacement) is positive when the particle moves in the positive x direction, and negative when the particle moves in the negative x direction.
  - It is not the same as distance travelled.
    - e.g. if the particle moves from point O to point P and then to point Q, the distance travelled is OP (30 m) + PQ (40 m) = 70 m.
    - The displacement is -10 – 0 = -10 m.
    - Displacement is numerically equal to the distance, when the particle is moving in the same direction. It is numerically less than the distance, when the particle changes direction.
    - Displacement can be any real number, whereas distance is non-negative.



Motion along a straight line - velocity

Average velocity $\bar{v}$ over time interval $[t_1, t_2]$ is defined as

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Represents how much the position changes (on an average), for a unit time increment.

Instantaneous velocity v at a time t is defined as

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

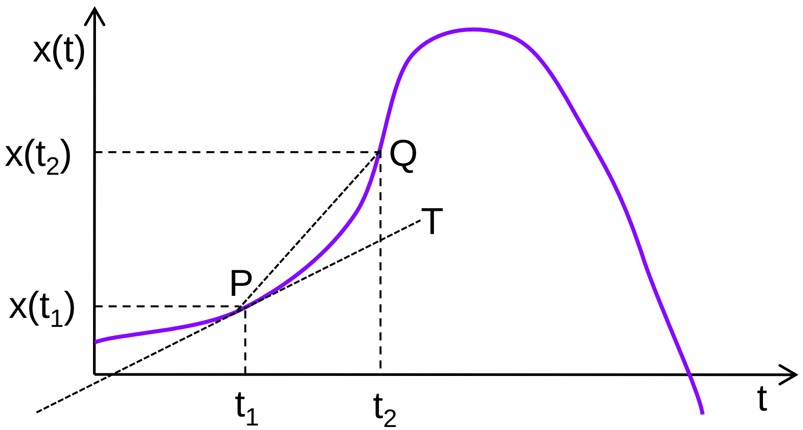
Represents how much the position changes per unit time, at the given time instant t.

Both $\bar{v}$ and v are positive for motion in the positive x direction, and negative for motion in the negative x direction. Unit of v is m/s (meters per sec).

|v| is the instantaneous speed of the particle.

In the graph of x(t),

- Average velocity over $[t_1, t_2]$ = slope of chord PQ
- Instantaneous velocity at t_1 = slope of tangent PT



Motion along a straight line - acceleration

Acceleration can be defined in terms of velocity, in the same way that velocity was defined in terms of position.

Average acceleration $\bar{a}$ over time interval $[t_1, t_2]$ is defined as

$$\bar{a} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Represents how much the velocity changes (on an average), for a unit time increment.

Instantaneous acceleration a at a time t is defined as

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt}$$

Represents how much the velocity changes per unit time, at the given time instant t.

Unit of a is (m/s)/s = m/s^2 (meters per second squared).

- Positive acceleration implies v is increasing. However, if v is negative, it implies that v is becoming less negative; that is the body (moving in the negative x direction) is slowing down.
- Therefore a body is speeding up, only when v and a have the same sign; and it is slowing down when v and a have opposite signs.

Straight line motion with uniform acceleration

Uniform acceleration implies a is constant, hence considering the interval [0, t]

$$a = \bar{a} = \frac{v(t) - v(0)}{t}$$

Denoting v(t) as v and v(0) as v_0, we can write

$$v = v_0 + at \qquad [1]$$

To find x, it is easiest to use the fact that $\frac{dx}{dt} = v = v_0 + at$

$\therefore x(t) = \int (v_0 + at)dt = v_0 t + \frac{1}{2}at^2 + C$ (where C is a constant to be determined)

By substituting t = 0 in the above, we find that constant C = x(0)

Denoting x(t) as x and x(0) as x_0, we have

$$x - x_0 = v_0 t + \frac{1}{2}at^2 = (v_0 + \frac{at}{2})t = \frac{v + v_0}{2}t \qquad [2]$$

Substituting t = $\frac{v - v_0}{a}$ (from Eq 1) in $x - x_0 = \frac{v + v_0}{2}t$, we have

$$x - x_0 = \frac{v^2 - v_0^{\ 2}}{2a} \rightarrow v^2 = v_0^2 + 2a(x - x_0) \qquad [3]$$

Velocity versus time graph

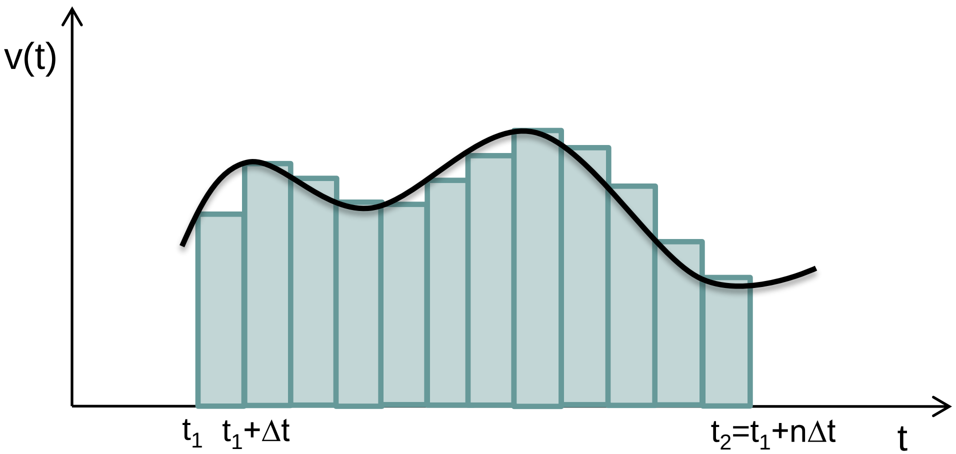
Since $\frac{dx}{dt} = v(t)$, we know from the Fundamental Theorem of Calculus that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)dt$$

The displacement over a time interval $[t_1, t_2]$ is the area under the v-t graph over that time interval. Area (in the sense used here) can also be negative, e.g. if v(t) is negative.

Same result can be derived from the v-t graph shown below. We divide the interval $[t_1, t_2]$ into n equal sub-intervals of width $\Delta t = (t_2 - t_1)/n$. When Δt is small enough, we can write

$x(t_1 + \Delta t) - x(t_1) = v(t_1)\Delta t$ [v(t) is almost constant in the interval Δt; hence $v(t)\Delta t$ is the displacement in this interval. It is the area of the 1st rectangle in the figure.]

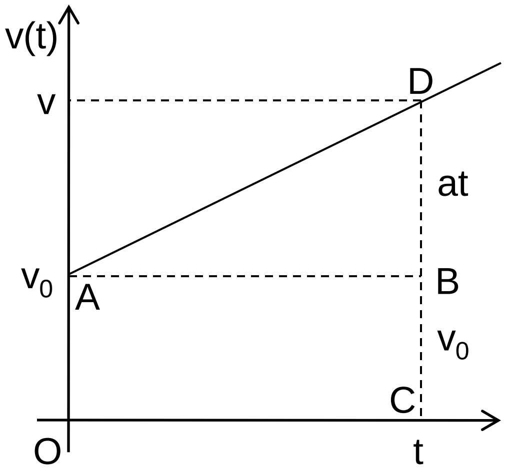


$x(t_1 + 2\Delta t) - x(t_1 + \Delta t) = v(t_1 + \Delta t)\Delta t$
$x(t_1 + 3\Delta t) - x(t_1 + 2\Delta t) = v(t_1 + 2\Delta t)\Delta t$
…
$x(t_1 + (n\text{-}1)\Delta t) - x(t + (n\text{-}2)\Delta t) = v(t_1 + (n\text{-}2)\Delta t)\Delta t$
$x(t_2) - x(t_1 + (n\text{-}1)\Delta t) = v(t_1 + (n\text{-}1)\Delta t)\Delta t$

Adding all we have
$x(t_2) - x(t_1)$ = sum of the area of all the rectangles.
This area becomes the area under the v-t graph over the interval $[t_1, t_2]$ when Δt approaches 0 (and hence n approaches infinity).

Velocity time graph for uniform acceleration

- From the v-t graph for uniform acceleration shown below, we see that
  $x(t) – x(0)$ = area of rectangle OABC + area of triangle ABD
  $= v_0 t + ½at^2$ = area of trapezium OADC $= ½(v + v_0)t$ as proved earlier.
- One important example of constant acceleration is **free fall** – a body falling down because of earth's gravitational force.
  - Magnitude of this acceleration is denoted by g, and is approximately 9.8 m/s^2, and its direction is vertically down.
  - This value of g holds, when the free fall is near the earth's surface (and the height of the fall is much less than the earth's radius), and we neglect air resistance.



Example of free fall (Example 1): A ball is thrown vertically up (at t = 0) with an initial speed v_0.

a. What is the maximum height reached by the ball?
b. When does it reach the maximum height?
c. When does it come back to the starting point?
d. What is the velocity when it comes back to the starting point?

Example 1 (free fall) solution

a) Take the positive x direction to be vertically up. Then the acceleration due to gravity is -g. We expect the ball to slow down as it goes up, and v = 0 at the maximum height. Then the ball starts to come down (and v is negative). We also take $x_0 = 0;$ that is the origin of the x axis is at the starting point. Using $v^2 = v_0^2 + 2ax$ with v = 0 and a = -g, we have

$0 = v_0^2 - 2gx \rightarrow x = \frac{v_0^2}{2g}$ (the maximum height reached by the ball).

b) Using $v = v_0 + at$ with v = 0 and a = -g, we have

$0 = v_0 - gt \rightarrow t = \frac{v_0}{g}$ (the time when it reaches the maximum height)

c) Using $x = v_0 t + \frac{1}{2}at^2$ with x = 0 and a = -g, we have

$0 = t(v_0 - \frac{gt}{2}) \rightarrow t = 0$ or $\frac{2v_0}{g}$ (t = 0 is the time its thrown up, so it comes back at t = $\frac{2v_0}{g}$)

The time to come back to the starting point, is twice the time to reach the maximum height.

d) Using $v = v_0 + at$ with t = $\frac{2v_0}{g}$ and a = -g, we have

$v = v_0 - g(\frac{2v_0}{g}) \rightarrow v = -v_0$ (comes back to the starting point, with negative of the initial velocity)

Example 2

The speed of a car going east is reduced from 30 to 25 m/s in a distance of 68.75m. Find

a) magnitude and direction of the acceleration, assuming it to be constant,

b) the time required to reduce the speed, and

c) the distance in which the car can be brought to rest from 25 m/s, assuming the acceleration in part a).

Solution: We will take the positive x direction towards the east. Also, we take x_0 as 0 - this just means we are positioning the origin at the initial position of the car (when t = 0).

a) Applying the equation $v^2 = v_0^2 + 2ax$, we have v = 25 m/s and v_0 = 30 m/s and x = 68.75m.

$$(25 \text{ m/s})^2 = (30 \text{ m/s})^2 + 2(a \text{ m/s}^2)(68.75 \text{ m})$$

$a = \dfrac{625 - 900}{137.5}$ = -2 m/s^2 (the negative sign implies the acceleration is towards the west).

The final answer must have the correct unit. This can be ensured by writing the units for each term in the calculations. This is a good practice, however it takes time, so use it whenever in doubt.

b) Using $v = v_0 + at$ with v = 25 m/s, v_0 = 30 m/s and a = -2 m/s^2, we have

$$25 = 30 - 2t \rightarrow t = \frac{5}{2} = 2.5s$$

c) Using $v^2 = v_0^2 + 2ax$ with v = 0, v_0 = 25 m/s and a = -2 m/s^2, we have

$$0 = 25^2 + 2 \times -2 \times x \rightarrow x = \frac{625}{4} = 156m$$

Example 3

A ball dropped from the top of a building takes $1/7$ s to pass a window 3.1m high. How far is the top of the window below the top of the building?

Take g = 9.8 m/s^2.

Solution: We will take the downward direction as the positive x direction.

Applying the equation $x = v_0 t + \frac{1}{2}at^2$ with x = 3.1m, t = $\frac{1}{7}s$ and a = 9.8 m/s^2, we have

$$3.1 = \frac{v_0}{7} + \frac{9.8}{2} \times \frac{1}{49}$$

$v_0 = (3.1 - 0.1)7 = 21$ m/s (the velocity when the ball reaches the window top)

Next we apply $v^2 = v_0^2 + 2ax$ with v_0 = 0, v = 21 m/s and a = 9.8 m/s^2 to find x, the distance of the window top from the building top.

$$21^2 = 2 \times 9.8x \rightarrow x = 22.5m$$

Example 4

[IIT 1995] Choose the correct answer:

A particle initially (i.e. at t = 0) moving with a velocity u is subjected to a retarding force, as a result of which it deccelerates at a rate

$a = -k\sqrt{v}$

where v is the instantaneous velocity and k is a positive constant. The time T taken by the particle to come to rest is given by

a) T = $\frac{2\sqrt{u}}{k}$ b) T = $\frac{2u}{k}$ c) T = $\frac{2u^{3/2}}{k}$ d) T = $\frac{2u^2}{k}$

We need to find out when velocity v = 0. What is given is $a = \frac{dv}{dt} = -k\sqrt{v}$ and v(0) = u.

We integrate both sides w.r.t t after writing the equation as $\frac{1}{\sqrt{v}}\frac{dv}{dt} = -k$

$\therefore 2\sqrt{v} = -kt + C$. Using the initial condition, we have C = $2\sqrt{u}$.

So $kt = 2\sqrt{u} - 2\sqrt{v}$. Therefore, when v is 0, t = $\frac{2\sqrt{u}}{k}$. The answer is a).

Motion in 3 dimensional space

- Definitions in 3D are similar to those for motion in a straight line, but it has to be in terms of the complete position vector **r**(t) = x(t)**i** + y(t)**j** + z(t)**k**.
- For time $t_1 < t_2$, $\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$ is called the **displacement** of the particle in the time interval $\Delta t = t_2 - t_1$. It is the change in the position of the particle.

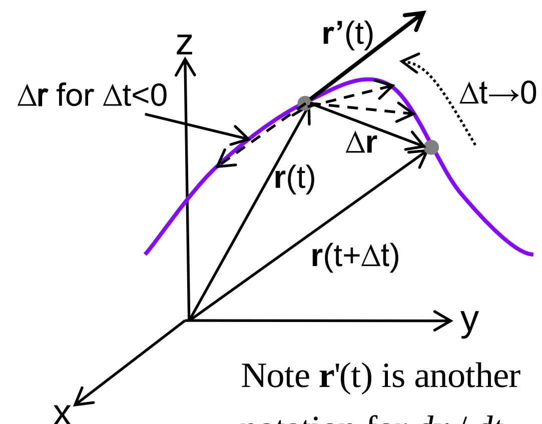
Average velocity $\bar{\mathbf{v}}$ over time interval $[t_1, t_2]$ is defined as $\bar{\mathbf{v}} = \dfrac{\mathbf{r}(t_2)-\mathbf{r}(t_1)}{t_2 - t_1}$

Instantaneous velocity **v** at a time t is defined as

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$

|**v**| is the instantaneous speed of the particle.

v is tangential to the path, and points in the direction of motion.



Note **r**'(t) is another notation for $d\mathbf{r}/dt$.

Average acceleration $\bar{\mathbf{a}}$ over time interval $[t_1, t_2]$ is defined as

$$\bar{\mathbf{a}} = \frac{\mathbf{v}(t_2)-\mathbf{v}(t_1)}{t_2 - t_1}$$

Instantaneous acceleration **a** at a time t is defined as

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t)-\mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k}$$

Motion in 3 dimensional space (continued)

Note the following:

x component of acceleration $a_x(t) = \frac{dv_x}{dt}$ and x component of velocity $v_x(t) = \frac{dx}{dt}$

Similarly for y and z components. Thus there is no cross dependence between the components, and each component can be treated separately.

For uniform acceleration $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, all components of **a** are constants.

Since each component can be treated separately, the equations derived earlier for motion in a straight line with uniform acceleration, can be applied to each component.

$$e.g. \quad v_x = v_{0x} + a_x t \qquad x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2 \qquad v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$$

where we have used the x components of **a** and **v** in the above equations.

Example 5

[IIT 1982] Choose the correct answer.

A particle is moving eastward with a velocity 5 m/s. In 10 s, the velocity changes to 5 m/s northwards. The average acceleration in this time is

a) 0

b) $\frac{1}{\sqrt{2}} m/s^2$ towards north-east

c) $\frac{1}{\sqrt{2}} m/s^2$ towards north-west

d) $\sqrt{2}\, m/s^2$ towards north-east

Solution: Take the positive x direction towards east and positive y direction towards north.

Then initial velocity $\mathbf{v}_1 = 5\mathbf{i}\, m/s$ and final velocity $\mathbf{v}_2 = 5\mathbf{j}\, m/s$

Average acceleration $\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t} = \frac{5\mathbf{j} - 5\mathbf{i}}{10} = \frac{\mathbf{j} - \mathbf{i}}{2} m/s^2$

So $|\bar{\mathbf{a}}| = \frac{1}{\sqrt{2}} m/s^2$ and direction is that of $\mathbf{j} - \mathbf{i}$ (north-west). Hence the answer is c)