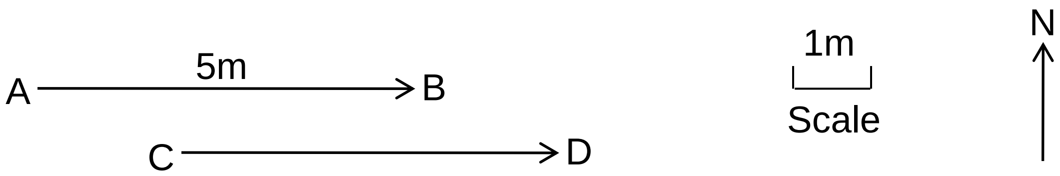
Lesson 14: Vectors (Part 1)

Vectors versus Scalars

- Physical quantities like height, mass, temperature etc. can be measured by specifying their magnitude (how large or small they are) in an appropriate unit. e.g.
  - Height of a person is 165 cm.
  - Temperature of water in a given bowl is 40^{o} C.
- Such quantites are called **scalars**.
- Physical quantities like force, displacement, velocity etc. need both a magnitude and a direction to be specified. e.g.
  - Displacement is the change in the position of a moving body. To specify the displacement over a time interval, we need to state how much the body has moved (its magnitude, e.g. 1 km) and in what direction it has moved (e.g. north or east).
  - To specify a force, we need to say how big it is (say 10 newtons), and in what direction it acts (a push sideways is different from a push to the front).
- Such quantities are called **vectors**.

Specifying vectors geometrically

- A vector is represented geometrically by a **directed line segment**: a line segment whose length is equal to the magnitude of the vector, and whose direction is in the direction of the vector. e.g.
  - a displacement of 5m to the east (a vector) can be represented by a line segment AB of length 5 cm (scale of 1:100) and pointing to the right as shown below.



- Moving the line segment AB parallel to itself doesn't change the vector that it represents. Thus line segment CD represents the same vector (same displacement) as AB.
- A is called the **initial point** of the directed line segment, and B is called the **terminal point**. The vector's direction is from the initial point to the terminal point.

The vector is written $\mathbf{a} = \vec{a} = \overrightarrow{AB} = \overrightarrow{CD}$

Boldface letters like $\mathbf{a}, \mathbf{b}$ are used in print to represent vectors. While writing, we represent the same as $\vec{a}, \vec{b}$.

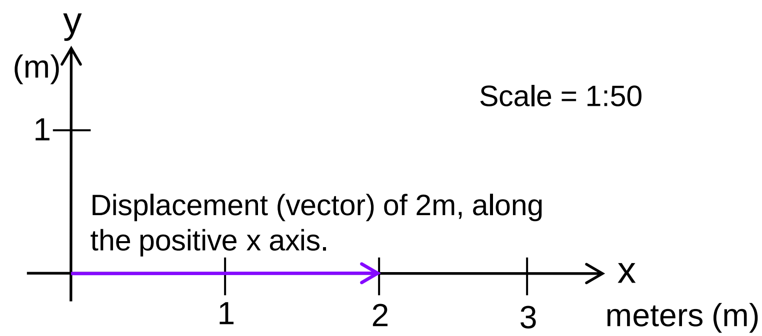
The directed line segment representing the vector is written as $\overrightarrow{AB},\ \overrightarrow{CD}$; with the initial point stated first.

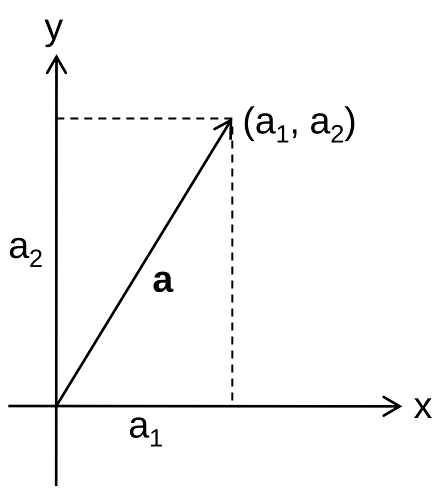
The **magnitude** of vector **a**, given by the length of the directed line segment, is written as

$\left|\overrightarrow{AB}\right|$, AB, $|\mathbf{a}|$, $|\vec{a}|$ or a.

Specifying vectors algebraically

- To manipulate vectors, an algebraic expression is better. Given a vector **a**, we do this as follows:
  - Setup a coordinate system (x – y axis) using the same scale on both axes.
    - e.g. to represent force, 1 cm on both axes can represent a force magnitude of 1N (newton). So successive integer values on the axes (representing force in newtons) are 1 cm apart.
    - Figure shows how a displacement of 2m along the +x axis, can be represented on paper.
  - Since vectors of the same length and direction (which are also parallel) are equal, move vector **a**'s initial point to the origin of the coordinate system. This is called the standard position for the vector.



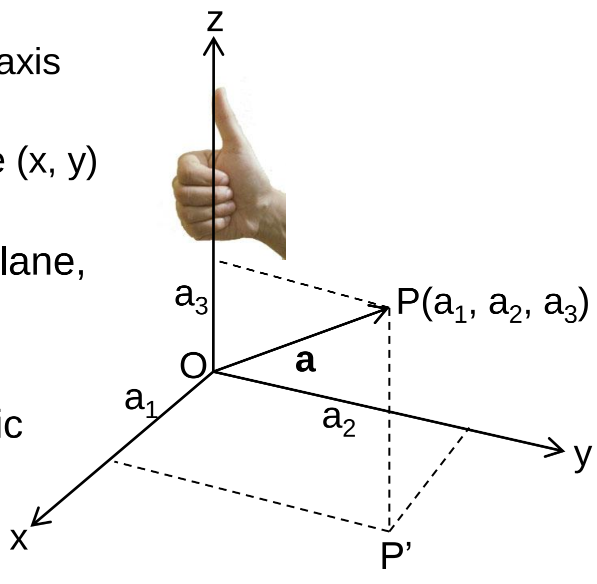


- The coordinates of the terminal point depends on the magnitude and direction of **a** and is the equivalent algebraic representation of **a**.
- We write **a** = $<a_1, a_2>$ and a_1 and a_2 are called the x and y **components** of **a**.

Specifying vectors algebraically in 3 dimensions

- A vector can point in any direction in space. To represent such a vector algebraically:
  - we add a z axis to the x-y coordinate system.
  - x, y and z axes meet at a common point (the origin O of the coordinate system) and are mutually perpendicular. They form a **right handed system**: implies the thumb points in the direction of z axis, when the fingers of the right hand curl to “push” the x axis towards the y axis.
  - The coordinates of a point P in space, are the numbers at which planes through P perpendicular to the axes, cut the axes.

- Another way to give the coordinates (x, y, z) of P are:
  - Draw a perpendicular from P to the z axis. The number on the z axis where it meets, is the z coordinate.
  - Draw a perpendicular from P to the xy plane meeting it at P’. The (x, y) coordinates of P are the (x, y) coordinates of P’.
- Note z = 0 represents all points in the xy plane, y = 0 is the xz plane, and x = 0 is the yz plane (also called **coordinate planes**).
- As in 2D case, move the vector **a** to the standard position.
- The coordinates of the terminal point (P in figure) is the algebraic representation of **a**.
- We write **a** = $\langle a_1, a_2, a_3 \rangle$ where a_1, a_2 and a_3 are called the components of **a**.



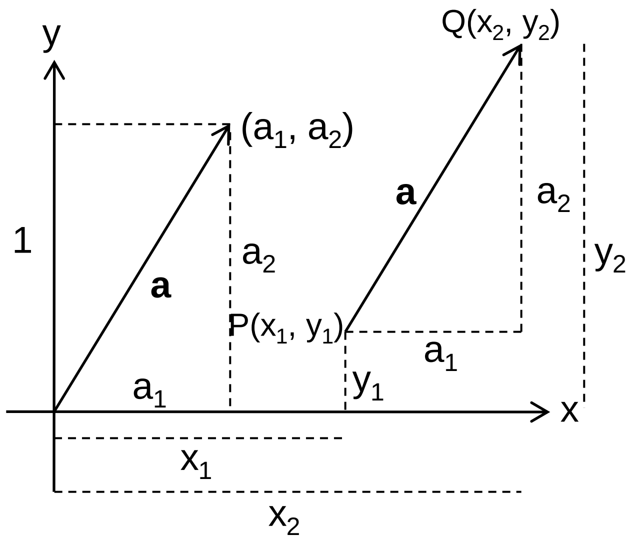
Vector components (continued)

- If vector **a** = $\langle a_1, a_2, a_3 \rangle$ has initial point as $P(x_1, y_1, z_1)$ instead of the origin, then its terminal point $Q(x_2, y_2, z_2)$ would be:
  - $x_2 = x_1 + a_1$, $y_2 = y_1 + a_2$, $z_2 = z_1 + a_3$ as shown below for 2D case (Fig 1).
  - Therefore, $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$, i.e. the vector from point P to point Q in component form is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
  - Length (magnitude) of **a** = length of PQ.
    - To find this, draw a rectangular box with P and Q as opposite corners, and faces parallel to the coordinate planes (Fig 2). Applying the Pythagoras theorem twice:

$$PQ^2 = PS^2 + SQ^2 = PR^2 + RS^2 + SQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$\text{So } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Fig 1



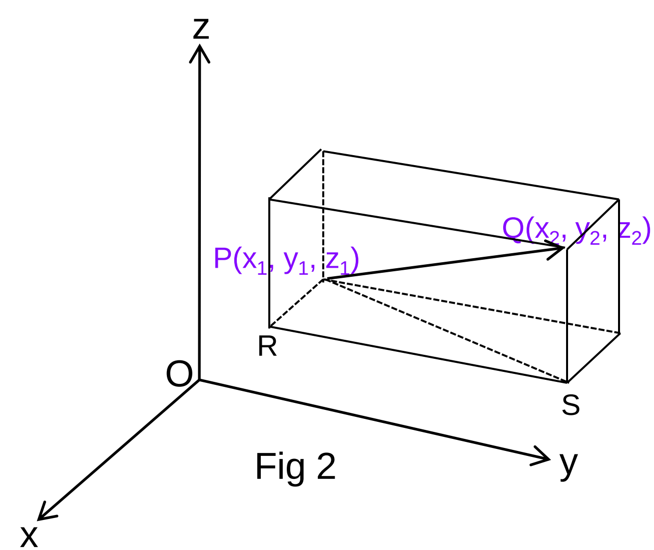


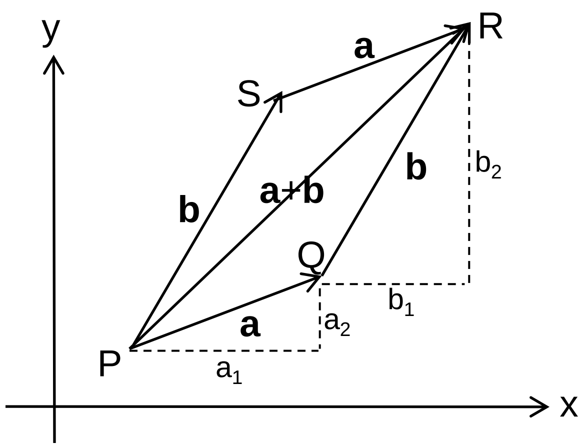
Fig 2

Vector addition

- Consider a moving body whose position changes from P to Q (represented by vector **a**) and then from Q to R (represented by vector **b**). Its total (net) displacement from the initial position P is given by the vector from P to R. So it seems reasonable to define vector addition as follows:

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{a} + \mathbf{b}$$

  - To add a vector **b** to vector **a**, we place the initial point of **b** at the terminal point of **a**, and the sum vector is from the initial point of **a** to the terminal point of **b**.
  - Note PQRS is a parallelogram since opposite sides represent the same vector (hence are parallel and have the same length). This shows **a + b = b + a** (vector addition is commutative).



- Vector addition is also referred to as the **parallelogram law of vector addition**.
  - With adjacent sides of the parallelogram representing vectors to be added, the diagonal is the sum.
- In component form, if **a** = <a_1, a_2, a_3> and **b** = <b_1, b_2, b_3> , then their sum **a** + **b** = <$a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$> as shown in the figure for 2D scenario.

Multiplying a vector by a scalar and vector subtraction

- If **a** is a vector and k is a scalar (real number), then k**a** is <u>defined</u> to be a vector whose magnitude is |k||**a**| and whose direction is the same as **a** if k is positive, and opposite to **a** if k is negative.
  - Vector k**a** is always parallel to **a**.
  - When k = −1, (−1)**a** = −**a** is the negative of **a**, and has the same magnitude as **a** but the opposite direction.
  - **a** + (−**a**) = **0** = <0, 0, 0>, the **zero vector**.
- In component form, if **a** = $\langle a_1, a_2, a_3 \rangle$, then $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
  - Shown for 2D case in Fig 1. Note PQR and P'Q'R' are similar triangles. Hence if P'Q':PQ = k, then the component ratios Q'R':QR and P'R':PR is also k.

If points A, B and C lie on a straight line (collinear), then $\overrightarrow{AB} = k\overrightarrow{AC}$ (where k is a constant). So when one vector is a scalar multiple of another, we also call them as collinear vectors.

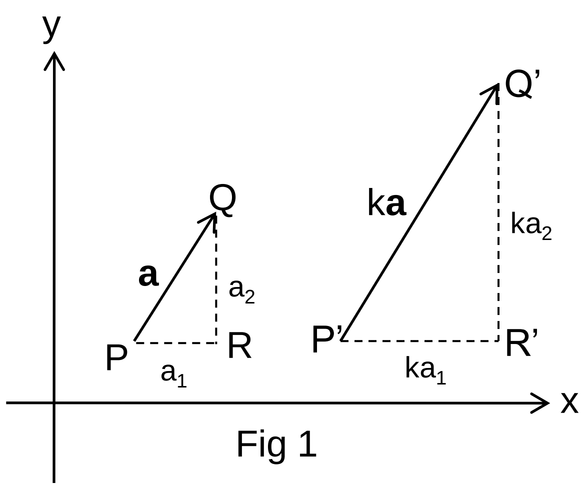


Fig 1

- If **a** and **b** are vectors, then **a** – **b** is <u>defined</u> as **a** + (−**b**).
  - With initial points of **a** and **b** placed together (Fig 2), **a** – **b** is a vector from the terminal point of **b** to the terminal point of **a**.
- In component form $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

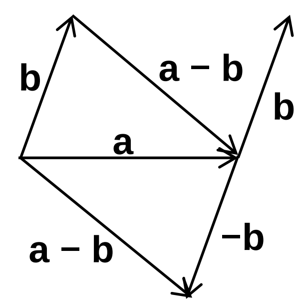
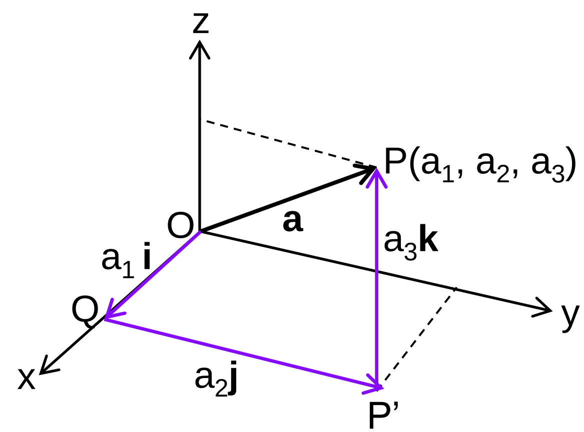


Fig 2

Properties of vector addition, scalar multiplication and the basis vectors

- If **a**, **b** and **c** are vectors and m and n are scalars, then the following properties can be readily verified using the component form of vector addition and vector multiplication by a scalar.
  - (**a** + **b**) + **c** = **a** + (**b** + **c**) [already seen that vector addition is commutative]
  - m(n**a**) = (mn)**a**
  - (m + n)**a** = m**a** + n**a**
  - m(**a** + **b**) = m**a** + m**b**
- The vectors **i** = <1, 0, 0>, **j** = <0, 1, 0> and **k** = <0, 0, 1> are called the **standard basis vectors**.
  - They are **unit vectors** (of length 1) and are directed along the positive x, y and z respectively.

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



The diagram shows the above representation explicitly. Note that

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP'} + \overrightarrow{P'P} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Basis vectors (continued) and example

Similarly, vector $\overrightarrow{\mathrm{PQ}}$ from $P(x_1,\ y_1,\ z_1)$ to $Q(x_2,\ y_2,\ z_2)$ is

$$\overrightarrow{\mathrm{PQ}} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

If O is the origin and point P = $(x_1,\ y_1,\ z_1)$, then

$\overrightarrow{\mathrm{OP}} = \langle x_1,\ y_1,\ z_1 \rangle = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ is called the **position vector** of point P.

Note $\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OP}}$.

Example: If $\overrightarrow{\mathrm{AB}} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and A is the point $(2, 1, -4)$, find point B and length AB.

$$AB = \sqrt{4^2 + (-2)^2 + 3^2} = \sqrt{29}$$

If B is the point $(x,\ y,\ z)$

$$\overrightarrow{AB} = (x-2)\mathbf{i} + (y-1)\mathbf{j} + (z+4)\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

Equating the components, we have

$$x - 2 = 4 \rightarrow x = 6$$

$$y - 1 = -2 \rightarrow y = -1$$

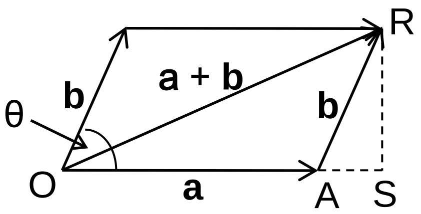
$$z + 4 = 3 \rightarrow z = -1$$

So B is $(6, -1, -1)$.

Magnitude and direction of sum vector

Given two vectors **a** and **b** that make an angle θ with each other (see figure), what is the magnitude OR of the sum vector **a** + **b**? Also, what is the angle α that the sum vector makes with vector **a**?

This is a standard result, which you should recall by remembering the (simple) derivation.



In ΔOAR, ∠OAR = $\pi - \theta$, hence applying the cosine rule, we have

$$OR^2 = a^2 + b^2 - 2ab\cos(\pi - \theta) \rightarrow OR^2 = a^2 + b^2 + 2ab\cos\theta$$

We could derive the same result using the right angled triangle OSR.

$$OR^2 = (OA + AS)^2 + RS^2 = (a + b\cos\theta)^2 + b^2\sin^2\theta = a^2 + b^2 + 2ab\cos\theta$$

$$\tan\alpha = \tan\angle AOR = \frac{RS}{OS} = \frac{b\sin\theta}{OA + AS} = \frac{b\sin\theta}{a + b\cos\theta}$$

Scalar or dot product of vectors

- When two vectors **a** and **b** are placed with their initial points coinciding, the dot product denoted as **a.b** is defined as |**a**||**b**|cosθ, where $0 \le \theta \le \pi$ is the angle between vectors **a** and **b**. Note the dot product is a number (scalar).
  - θ = 0 when **a** and **b** have the same direction, and θ = π when **a** and **b** have opposite directions.
  - When θ = π/2, the vectors are perpendicular (**orthogonal**) to each other and **a.b** = 0**.**
  - **a.b = b.a** : Dot product is commutative based on its definition.
  - If k is a scalar (real number), (k**a**).**b** = **a**.(k**b**) = k(**a.b**). Follows from definition of dot product and multiplication of a vector by a scalar.
  - **a.b** = OB.OP = |**b**|.(|**a**|cosθ) (see Fig 1). |**a**|cosθ is called the component of **a** in the direction of **b** (positive when θ < π/2 and negative when θ > π/2); while |**b**|cosθ is the component of **b** in the direction of **a.**
  - **a**.(**b** + **c**) = **a.b** + **a.c** (distributive property)
    - Shown in Fig 2, when all vectors are in the same plane.
    - **a**.(**b** + **c**) = a.OQ = a.(OP + PQ) = a.OP + a.PQ = **a.b** + **a.c**

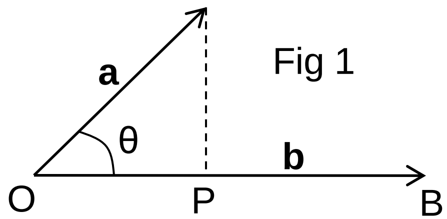


Fig 1

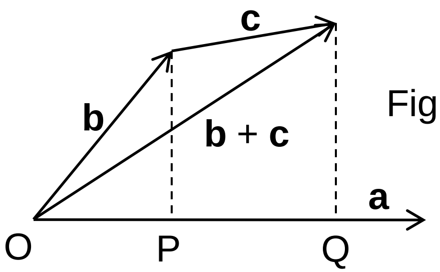


Fig 2

Scalar product in component form

- By definition of scalar product
  - **i.i** = **j.j** = **k.k** = 1
  - **i.j** = **j.k** = **k.i** = 0
- Using the above and the distributive property, if **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**, then **a.b** = $a_1b_1 + a_2b_2 + a_3b_3$ (terms like a_1b_2**i.j**, a_1b_3**i.k** etc. are 0).

Note: NCERT text calls the component of **a** in the direction of **b**, as the (scalar) *projection* of **a** on **b**. We can write this as:

$$\text{Projection of } \mathbf{a} \text{ on } \mathbf{b} = |\mathbf{a}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}\cdot\widehat{\mathbf{b}} \text{ where } \widehat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|} \text{ is the unit vector in the direction of } \mathbf{b}.$$

$$\text{Also } \left(|\mathbf{a}|\cos\theta\right)\widehat{\mathbf{b}} = \frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{b}|^2}\mathbf{b} \text{ is called the vector projection of } \mathbf{a} \text{ on } \mathbf{b} \text{ (vector } \overrightarrow{OP} \text{ in Fig 1 of last page).}$$

Note a vector with a hat (^) symbol (such as $\widehat{\mathbf{b}}$ above), is usually a unit vector.

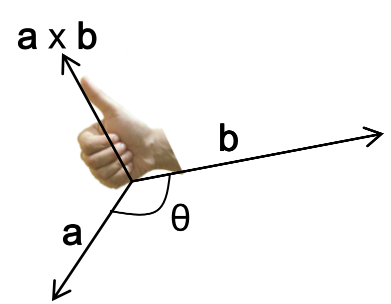
$$\text{Example: If } \mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \text{ then}$$

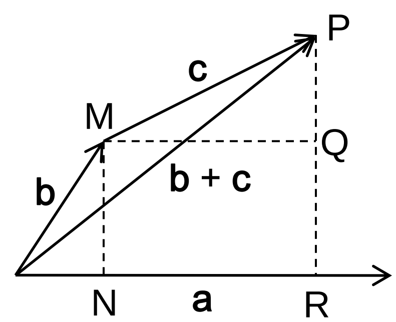
$$\mathbf{a.b} = \; 2.3 + -3.2 + 4.-1 = -4$$

$$\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-4}{\sqrt{29}\sqrt{14}} \rightarrow \theta \cong 101^\circ$$

Vector or cross product

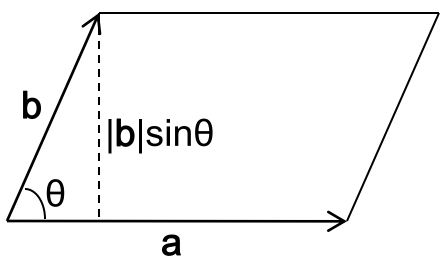
- When two vectors **a** and **b** are placed with their initial points coinciding, then their **cross** (or **vector**) **product** written as $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{n}$.
  - where **n** is a unit vector perpendicular to the plane of **a** and **b**, such that **a**, **b** and **n** form a right handed system (i.e. if the fingers of the right hand curl to "push" **a** into **b**, the thumb points in the direction of **n**.
  - $0 \leq \theta \leq \pi$ is the angle between the vectors **a** and **b**.
  - Note the cross product is a vector (unlike the dot product).
- $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ since **n** reverses direction. Hence cross product is not commutative.
- If k and l are scalars, $k\mathbf{a} \times l\mathbf{b} = (kl)(\mathbf{a} \times \mathbf{b})$: Follows from the definition of cross product, and multiplication of a vector with a scalar.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive property)
  - Shown below when all vectors are in the same plane.
  - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = a.PR.\mathbf{n} = a.(RQ + QP).\mathbf{n} = a.NM.\mathbf{n} + a.QP.\mathbf{n} = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (**n** is a unit normal coming out of the page)



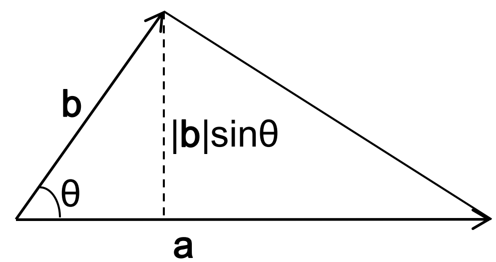


Geometric interpretation to cross product

- |**a** x **b**| is the area of the parallelogram with adjacent sides as vectors **a** and **b** (see figure).
- (1/2) |**a** x **b**| is the area of the triangle with adjacent sides as vectors **a** and **b** (see figure).



$$\text{Area} = \text{base x height}$$
$$= |\mathbf{a}|\times|\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|$$



$$\text{Area} = \frac{1}{2}(\text{base x height})$$
$$= \frac{1}{2}|\mathbf{a}|\times|\mathbf{b}|\sin\theta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

Example: Show that $|\mathbf{a} \times \mathbf{b}|^2 = (\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{b}) - (\mathbf{a}.\mathbf{b})(\mathbf{a}.\mathbf{b})$

Solution: Case 1: If **b** = k**a**, then the left side is 0, and the right side is $(a^2)(k^2a^2) - (ka^2)(ka^2) = 0$.

Case 2: If **a** and **b** are not parallel, we can consider them as adjacent sides of a parallelogram, and the left side is A^2, square of the parallelogram area.

$A = |ab\sin\theta| \rightarrow A^2 = a^2b^2(1 - \cos^2\theta) = (\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{b}) - (\mathbf{a}.\mathbf{b})(\mathbf{a}.\mathbf{b})$ (the right side).

Cross product in component form

- By definition of cross product
  - **i** x **i** = **j** x **j** = **k** x **k** = 0
  - **i** x **j** = −(**j** x **i**) = **k**
  - **j** x **k** = −(**k** x **j**) = **i**
  - **k** x **i** = −(**i** x **k**) = **j**
  - Fig 1 should help you recall the above quickly. When we move in the direction of the arrows, for example **i** x **j** gives **k** with a plus sign. Moving in the opposite direction, for example, **k** x **j** gives **i** with a minus sign.
- Based on the above and distributive property for cross product, if **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**, then

$$\mathbf{a}\times\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

$$= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k}) + a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j}) + a_2b_3(\mathbf{j}\times\mathbf{k}) +$$

$$a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k})$$

$$= 0 + a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + 0 + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} + 0$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$\therefore \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{(in determinant form)}$$

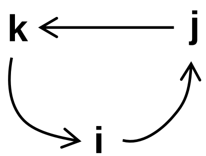


Fig 1: Remembering cross products between unit vectors

Examples

We will cover determinants in a later lesson; but for now, note that if **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k**, then the coefficients of **i**, **j** and **k** in **a** x **b**, are the denominators from the "rule of cross multiplication", when we treat **i**, **j** and **k** as variables (with vector **a** listed first).

So **a** x **b** = $(a_2b_3 - a_3b_2)$**i** + $(a_3b_1 - a_1b_3)$**j** + $(a_1b_2 - a_2b_1)$**k**

If one of the vectors has only one component, it is easier to multiply using the distributive property of the cross product.

Question: Let point $\mathrm{P} = (0,\ 0,\ 0),\ \mathrm{Q} = (1,\ 0,\ 0)$ and $\mathrm{R} = (2,\ 1,\ 1)$

Find area of triangle PQR and the unit normal to the plane of P, Q and R.

Solution: Area of triangle PQR = $\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$

$\overrightarrow{PQ} = \mathbf{i}$ and $\overrightarrow{\mathrm{PR}} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \times (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{k} - \mathbf{j}$

$\therefore Area = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$

$\overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane of P, Q and R. So unit normal is $\mathbf{n} = \frac{\mathbf{k} - \mathbf{j}}{\sqrt{2}}$ or $-\mathbf{n} = \frac{\mathbf{j} - \mathbf{k}}{\sqrt{2}}$

Examples (continued)

[IIT 1981, True or False]: Let **A**, **B** and **C** be unit vectors. Suppose that **A.B** = **A.C** = 0, and the angle between **B** and **C** is 30°. Then **A** = ±2(**B** x **C**).

Solution: Note vectors **B** and **C** define a plane (since they are not parallel), and vector **A** is perpendicular to this plane (since **A.B** = **A.C** = 0). So vector **A** has the same or opposite direction as **B** x **C**.

|**B** x **C**| = 1 x 1 x sin(30°) = 1/2. Since **A** is a unit vector, the statement **A** = ±2(**B** x **C**) is true.

[IIT 1994] Let α, β, γ be distinct real numbers. The points with position vectors α**i** + β**j** + γ**k**, β**i** + γ**j** + α**k**, γ**i** + α**j** + β**k**

a) are collinear
b) form an equilateral triangle
c) form a scalene triangle
d) form a right angled triangle

Solution: Let the position vectors $\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}, \beta\mathbf{i}+\gamma\mathbf{j}+\alpha\mathbf{k}, \gamma\mathbf{i}+\alpha\mathbf{j}+\beta\mathbf{k}$ correspond to points A, B and C respectively. We then have

$$\overrightarrow{AB}=(\beta-\alpha)\mathbf{i}+(\gamma-\beta)\mathbf{j}+(\alpha-\gamma)\mathbf{k},\quad \overrightarrow{AC}=(\gamma-\alpha)\mathbf{i}+(\alpha-\beta)\mathbf{j}+(\beta-\gamma)\mathbf{k}$$

and $\overrightarrow{BC}=(\gamma-\beta)\mathbf{i}+(\alpha-\gamma)\mathbf{j}+(\beta-\alpha)\mathbf{k}$

Note all of the them have a magnitude of $\sqrt{(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2}$.

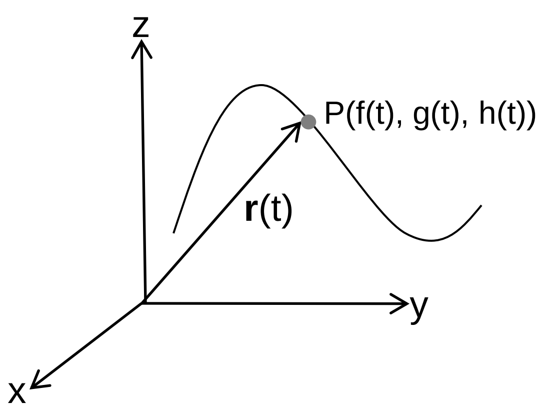
Hence the points form an equilateral triangle (option b) is correct).

We have covered the basics of vectors, however to make this useful from a physics viewpoint, we will now consider the calculus of vector valued functions of a real variable.

Make sure that you have read our calculus introduction, before reading the following material.

Parametric curves in 3 dimensions and vector functions

- Parametric curves in 2D was considered in the derivatives lesson.
- When a particle moves in space, we can consider its coordinates (x, y, z) as functions of time t (the parameter). The set of possible values of t is called the parameter interval (denoted I here).
- Let $x = f(t)$, $y = g(t)$ and $z = h(t)$. These are called the parametric equations of the curve traced out (in 3D space), when t increases in I. The curve is called a parametric curve, and it is the path of the moving particle.
- We can also consider the curve as being traced out by the vector $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$; with the vector drawn from the origin of the coordinate system. This vector is called the **position vector** of the moving particle.
  - As t increases in I, the terminal point P of **r** traces the path of the moving particle.
  - **r**(t) is called a **vector function** of t. The domain of the function is the set of allowed values for t, and its range is the set of corresponding values for vector **r**.



Limit and continuity of vector functions

- Calculus concepts applied to real functions of a real variable (also called scalar functions) can be applied to vector functions as well.
- Thus a vector function **u**(t) approaches a limit vector **L** as t approaches t_0, if |**u**(t) – **L**| (length of the difference vector) can be made as small as we please, for all t sufficiently close to t_0.

If $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$ and $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then

$$\lim_{t\to t_0}\mathbf{u}(t) = \mathbf{L} \rightarrow \lim_{t\to t_0} u_1(t) = L_1;\ \lim_{t\to t_0} u_2(t) = L_2;\ \lim_{t\to t_0} u_3(t) = L_3$$

Limits of vector functions can be calculated one component at a time.

If $\lim_{t\to t_0}\mathbf{u}(t) = \mathbf{u}(t_0)$, then the vector function $\mathbf{u}(t)$ is continuous at the point $t = t_0$.

For $\mathbf{u}(t)$ to be continuous at $t = t_0$, each component of $\mathbf{u}$ must be continuous at $t = t_0$.

A function $\mathbf{u}(t)$ is continuous, if it is continuous at all points in its domain (i.e. for all valid values of t).

Derivative of a vector function

Derivative of a vector function **u**(t) is defined below (in a way similar to scalar functions):

$$\mathbf{u}'(\mathrm{t}) = \lim_{h\to 0} \frac{\mathbf{u}(\mathrm{t}+h)-\mathbf{u}(\mathrm{t})}{h}$$

If $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$, then

$$\mathbf{u}'(\mathrm{t}) = \lim_{h\to 0} \frac{[u_1(t+h)\mathbf{i} + u_2(t+h)\mathbf{j} + u_3(t+h)\mathbf{k}] - [u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}]}{h}$$

$$= \lim_{h\to 0} \frac{u_1(t+h)-u_1(t)}{h}\mathbf{i} + \lim_{h\to 0} \frac{u_2(t+h)-u_2(t)}{h}\mathbf{j} + \lim_{h\to 0} \frac{u_3(t+h)-u_3(t)}{h}\mathbf{k}$$

$$\mathbf{u}'(\mathrm{t}) = \frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k}$$

A vector function is differentiable at a point $t = t_0$ if and only if each of its component functions is differentiable at t_0.

If $t = g(s)$, then

$$\frac{d\mathbf{u}}{ds} = \frac{du_1}{ds}\mathbf{i} + \frac{du_2}{ds}\mathbf{j} + \frac{du_3}{ds}\mathbf{k}$$

By the chain rule for scalar functions $\dfrac{du_i}{ds} = \dfrac{du_i}{dt}\dfrac{dt}{ds}$ for i = 1, 2, 3

$$\frac{d\mathbf{u}}{ds} = \left(\frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k}\right)\frac{dt}{ds} = \mathbf{u}'(g(s))g'(s)$$

Derivatives (continued)

If **u**(t) and **v**(t) are differentiable vector functions, then

$$\frac{d(\mathbf{u}\cdot\mathbf{v})}{dt}=\frac{d\mathbf{u}}{dt}\cdot\mathbf{v}+\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}$$

Proof:

When t changes by Δt, let **u** change by Δ**u** and **v** change by Δ**v**

$$\Delta(\mathbf{u}\cdot\mathbf{v})=(\mathbf{u}+\Delta\mathbf{u})\cdot(\mathbf{v}+\Delta\mathbf{v})-\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\Delta\mathbf{v}+\Delta\mathbf{u}\cdot\mathbf{v}+\Delta\mathbf{u}\cdot\Delta\mathbf{v}$$

$$\frac{d(\mathbf{u}.\mathbf{v})}{dt}=\lim_{\Delta t\to 0}\left(\mathbf{u}\cdot\frac{\Delta\mathbf{v}}{\Delta \mathrm{t}}+\frac{\Delta\mathbf{u}}{\Delta \mathrm{t}}\cdot\mathbf{v}+\frac{\Delta\mathbf{u}}{\Delta \mathrm{t}}\cdot\Delta\mathbf{v}\right)=\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}+\frac{d\mathbf{u}}{dt}\cdot\mathbf{v}$$

(Because **v** is differentiable, $\Delta\mathbf{v}\to\mathbf{0}$ when $\Delta t\to 0,$ hence $\frac{\Delta\mathbf{u}}{\Delta \mathrm{t}}\cdot\Delta\mathbf{v}\ \to\ \frac{d\mathbf{u}}{dt}\cdot\mathbf{0}=0$)

In a similar way, if **u**(t) and **v**(t) are differentiable vector functions,

and f(t) is a differentiable scalar function, we can show that

$$\frac{d(\mathbf{u}\times\mathbf{v})}{dt}=\frac{d\mathbf{u}}{dt}\times\mathbf{v}+\mathbf{u}\times\frac{d\mathbf{v}}{dt}$$

$$\frac{d(f(t)\mathbf{u}(t))}{dt}=\frac{df}{dt}\mathbf{u}+f\frac{d\mathbf{u}}{dt}$$

Velocity and acceleration

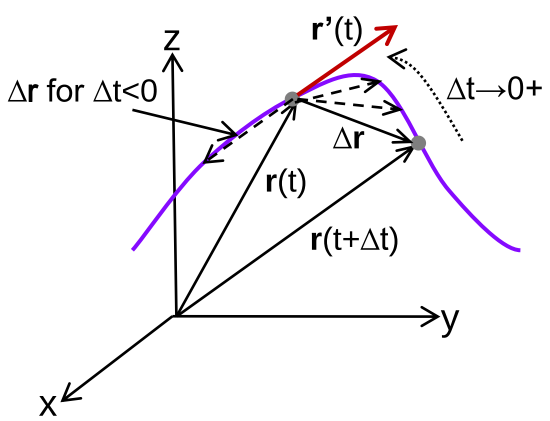
- If **r**(t) is the position vector of a moving particle (as a function of time t), then **r**'(t) has the following properties:
  - It is tangential to the path, and points in the direction of motion.
    - Note when Δt < 0, Δ**r** would be opposite to the direction of motion, but the ratio Δ**r**/Δt is still in the direction of motion.
  - Since |Δ**r**| is the distance travelled in Δt (when Δt is sufficiently small), |**r**'(t)| is the instantaneous speed.
  - Hence it is useful to define **r**'(t) as the velocity of the moving particle (rate of change of its position) .

$$\text{Velocity of a particle } \mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ where } \mathbf{r}(t) \text{ is its position vector}$$

$$\text{Particle speed} = |\mathbf{v}|$$

$$\text{Unit vector in the direction of motion} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$\text{Particle acceleration defined as } \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$



Integrals of vector functions

A differentiable function **U**(t) is an antiderivative of **u**(t) in an interval I,

if $\frac{d\mathbf{U}(t)}{dt} = \mathbf{u}(t)$ for all t in I.

If $\mathbf{U}(t) = U_1(t)\mathbf{i} + U_2(t)\mathbf{j} + U_3(t)\mathbf{k}$ and $u(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$

Then $\frac{dU_i}{dt} = u_i(t)$; and $U_i(t) + C_i$ (where C_i is a constant) is the set of all antiderivatives of u_i.

Hence $\mathbf{U}(t) + \mathbf{C}$, where **C** is a constant vector is the set of all antiderivatives of **u**(t). We write

$$\int \mathbf{u}(t)dt = \mathbf{U}(t) + \mathbf{C}$$

$$\int_a^b \mathbf{u}(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{u}(t_i)\Delta t \quad \text{where } \Delta t = \frac{b-a}{n} \text{ and } t_i = a + i\Delta t$$

$$= \lim_{n\to\infty} \left[\mathbf{i}\sum_{i=1}^{n} u_1(t_i)\Delta t + \mathbf{j}\sum_{i=1}^{n} u_2(t_i)\Delta t + \mathbf{k}\sum_{i=1}^{n} u_3(t_i)\Delta t \right]$$

$$= \mathbf{i}\int_a^b u_1(t)dt + \mathbf{j}\int_a^b u_2(t)dt + \mathbf{k}\int_a^b u_3(t)dt = \mathbf{i}U_1(t)\Big|_a^b + \mathbf{j}U_2(t)\Big|_a^b + \mathbf{k}U_3(t)\Big|_a^b$$

$$= \mathbf{U}(b) - \mathbf{U}(a)$$

Examples

Example: If $|\mathbf{u}(t)| = C$ (a constant), show $\mathbf{u}(t)\cdot\mathbf{u}'(t) = 0$

Solution: $|\mathbf{u}(t)|^2 = \mathbf{u}(t)\cdot\mathbf{u}(t) = C^2$

Differentiate both sides w.r.t. t, and use the derivative rule for a dot product.

$$\frac{d(\mathbf{u}(t)\cdot\mathbf{u}(t))}{dt} = \frac{d\mathbf{u}}{dt}\cdot\mathbf{u} + \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = \frac{d(C^2)}{dt} = 0$$

$$\therefore 2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0 \rightarrow \mathbf{u}\cdot\mathbf{u}' = 0$$

For uniform circular motion of a particle, **|v|** is a constant (where **v** is the velocity vector). Hence, **v**.d**v**/dt = 0, i.e. the acceleration is perpendicular to the velocity.

Example: $\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i}$. Find $\mathbf{r}(t)$.

Solution: Integrating with respect to t

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = 0 + \mathbf{C} = \mathbf{i} \rightarrow \mathbf{r}(t) = (t^2+1)\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k}$$