Lesson 10: Limits and Continuity

Limit of a function

- The concept of limit of a function is central to all other concepts in calculus (like continuity, derivative, definite integrals etc. as we will see later).
- To understand the concept, consider the function f(x) = x^2. What value does the function f "approach", when x "approaches" 2?
  It approaches $2^2 = 4$. What does this mean?
  - It means that the difference between f(x) and 4 can be made as small as we please, if the difference between x and 2 is made sufficiently small.
  - We say that the limit of the function x^2 as x approaches 2 is the value 4.
- In general, a function f(x) has a **limit** L as x approaches "a", if f(x) can be made as close to L as we like, for all x sufficiently close to a. We write this as:

$$\lim_{x \to a} f(x) = L$$

Example: Let $g(x) = 3x^2 + x + 1$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1}(3x^2 + x + 1) = 3.1^2 + 1 + 1 = 5$$

$$\lim_{x \to 3} g(x) = \lim_{x \to 3}(3x^2 + x + 1) = 3.3^2 + 3 + 1 = 31$$

Does it mean, that to evaluate the limit of f(x) as x approaches "a", we just evaluate f(a)? Not always (the limit is f(a) only for continuous functions, as we will see later). In fact, f(a) need <u>not</u> be defined for the limit to exist.

Limit of a function (continued)

Consider the function f(x) below (note a function can have different definitions, in different parts of its domain):

$$f(x) = x^2 \text{ for } x \neq 1, \text{ and}$$

$$f(x) = 2 \text{ for } x = 1$$

$$\lim_{x \to 1} f(x) = 1^2 = 1 \text{ but } f(1) = 2$$

To determine the limit of f(x) as x approaches “a”, we need to know the behavior of f(x) for x near “a”. In the above example, even if f(1) was undefined, the limit would still be 1. This is because the behavior of f(x) near 1, depends on the behavior of x^2 (the function definition near 1).

In general, we don’t define limit for a point x = a; instead we define it for “x approaches a”.

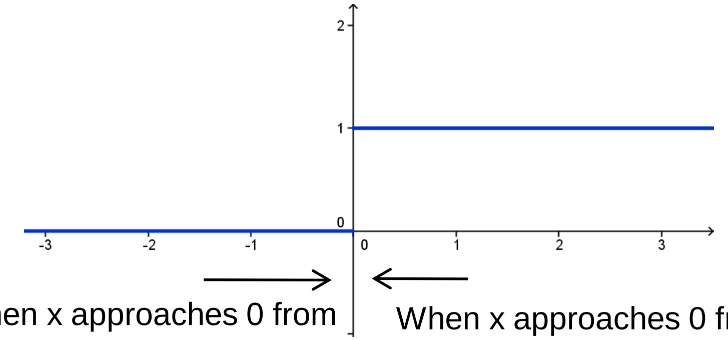
Consider the unit step function u(x) shown below. What is its limit as x approaches 0?

$$u(x) = 1 \text{ for } x \geq 0$$

$$= 0 \text{ for } x < 0$$

As you might have guessed, u(x) does not have a limit as x approaches 0.

When x approaches 0 from the right (values greater than 0), u(x) has the limit 1 (called the right hand limit); when x approaches 0 from the left (values less than 0), u(x) has the limit 0 (called the left hand limit).



When x approaches 0 from the left, u(x) is 0.

When x approaches 0 from the right, u(x) is 1

Right hand and Left hand limits

So by definition, f(x) has a **right hand limit** L as x approaches “a”, if f(x) can be made as close to L as we like, for all x greater than but sufficiently close to “a” (x approaches “a” from the right along the x axis). We write this as:

$$\lim_{x \to a^+} f(x) = L$$

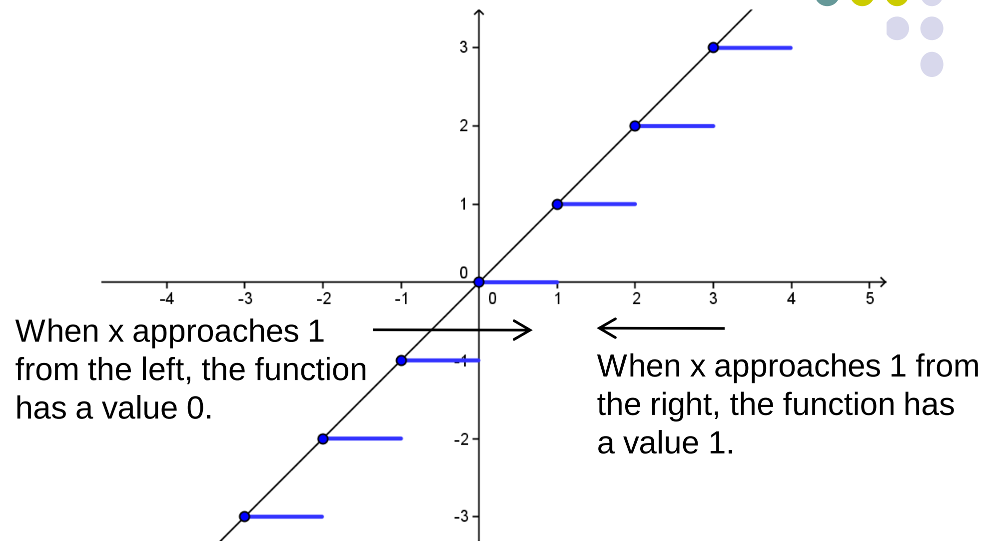
Similarly, f(x) has a **left hand limit** L as x approaches “a”, if f(x) can be made as close to L as we like, for all x less than but sufficiently close to “a” (x approaches “a” from the left along the x axis). We write this as:

$$\lim_{x \to a^-} f(x) = L$$

For f(x) to have a limit as x approaches “a”, both the right hand and left hand limit as x approaches “a” must exist and they must be equal. The unit step function u(x) does not have a limit when x approaches 0, since the right hand and left hand limit are different.

Scenarios when the limit does not exist

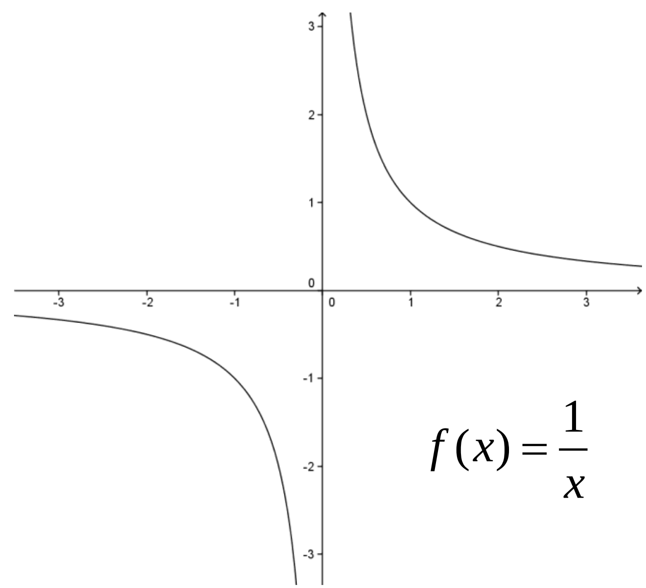
For the greatest integer function, the right and left hand limits are not equal (and hence the limit does not exist), when x approaches <u>any</u> integer value.



The function f(x) = 1/x approaches infinity as x approaches 0 from the right, and approaches minus infinity as x approaches 0 from the left.

Infinity is not a number, hence neither the right hand nor left hand limit exists when x approaches 0.

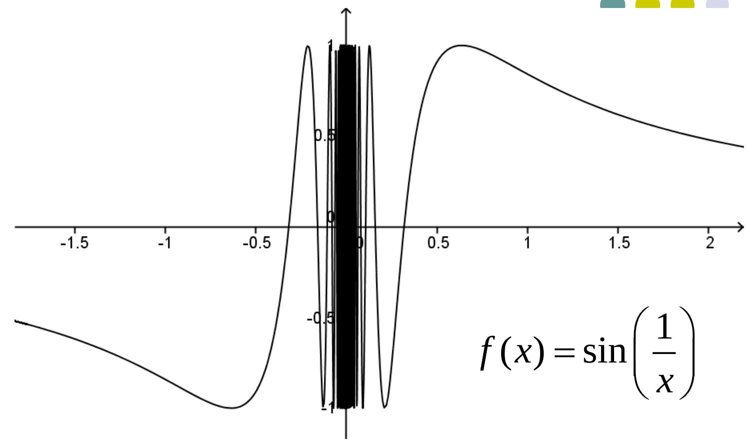
f(x) approaches infinity, means that f(x) can be made larger than any value we choose; for all values of x greater than but sufficiently close to 0.



Scenarios when the limit does not exist (continued), and using infinity in limits

A less common example of limit not existing is the function f(x) = sin(1/x) as x approaches 0. To understand this example, you must know how the sin function is defined for <u>all</u> values of x.

The function rapidly oscillates between −1 and +1 (as we come close to 0), and never approaches any specific value. So both the right hand and left hand limit does not exist when x approaches 0.



Though infinity is not a number, it is convenient to use it in expressions as shown below.

a) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

b) $\lim_{x \to \infty} \frac{1}{x} = 0$

Example a) is what we have already seen.

Example b) says that 1/x can be made as close to 0, as we please; by making x sufficiently large.

Limit rules

The limit rules can be used to determine the limit, when functions are combined via operations of addition, multiplication etc.

Limit rules

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then

$\lim_{x \to a} \left(f(x) + g(x) \right) = L + M$ (sum rule)

$\lim_{x \to a} \left(f(x) - g(x) \right) = L - M$ (difference rule)

$\lim_{x \to a} \left(f(x) g(x) \right) = L \times M$ (product rule)

$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ when $M \neq 0$ (quotient rule)

$\lim_{x \to a} \left(f(x) \right)^{r/s} = L^{r/s}$ where r and s are integers (power rule)

The above rules seem reasonable; for example, if f(x) approaches L and g(x) approaches M, as x approaches “a”, we expect the sum f(x) + g(x) to approach L + M, as x approaches “a”.

Some limit examples

$$\lim_{x \to a} x = a$$

$\lim_{x \to a} k = k$ where k is a constant

If k is a constant and n is a positive integer, then (applying the product rule)

$$\lim_{x \to a} kx^n = \lim_{x \to a} k \cdot \lim_{x \to a} \underbrace{x \cdot x \ldots \cdot x}_{\text{n times}} = k \cdot \underbrace{\lim_{x \to a} x \cdot \lim_{x \to a} x \ldots \cdot \lim_{x \to a} x}_{\text{n times}} = ka^n$$

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ (polynomial function), then (applying the sum rule, and the above result)

$$\lim_{x \to a} P(x) = \lim_{x \to a} a_n x^n + \lim_{x \to a} a_{n-1} x^{n-1} + \ldots + \lim_{x \to a} a_1 x + \lim_{x \to a} a_0$$

$$= a_n a^n + a_{n-1} a^{n-1} + \ldots + a_1 a + a_0 = P(a)$$

If P(x) and Q(x) are polynomial functions, then (using quotient rule and above result)

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)}$$ when $Q(a) \neq 0$

Note, ratio of two polynomial functions is called a rational function.

Some limit examples (continued)

If Q(a) = P(a) = 0, then (x – a) is a common factor, and we can cancel it out to evaluate the limit of the rational function P(x) / Q(x) as x approaches “a”. If Q(a) = 0, and P(a) ≠ 0, then the limit does not exist as x approaches “a”.

Example: Evaluate $\lim_{x\to 2}\frac{x^2-4}{x-2}$. Both x^2-4 and $x-2$ become zero at $x=2$ (called the 0/0 form).

But the limit can be written as $\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$

For $x\neq 2$, we can cancel $(x-2)$ and write $\lim_{x\to 2}(x+2)=4$. So the limit is 4.

In case, you are thinking why we added the condition x ≠ 2, then consider the definition of division. a/b by definition is ab^{-1}, where b^{-1} is the multiplicative inverse of b. A number x is the multiplicative inverse of y (and vice versa) if xy = 1. All real numbers except 0, have a multiplicative inverse; hence division by zero is undefined.

In the above example, division by (x – 2) is undefined when x = 2, so we add the condition. From a limit viewpoint also, exclusion of x = 2 makes sense, because we are interested in the behavior of the function <u>near</u> x = 2.

Some limit examples (continued)

Example: Show that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (where n is a positive integer).

Solution : We could divide $x^n - a^n$ by $x - a$ to get

$$\left(x^n - a^n\right) \big/ \left(x - a\right) = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1} \text{ (for } x \neq a)$$

Another way is to treat the right side as a geometric series with first term x^{n-1} and ratio a / x.

We then have $x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1} = \frac{x^{n-1}\left[1 - \left(a/x\right)^n\right]}{1 - a/x} = \frac{x^n - a^n}{x - a}$

$$\therefore \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a}\left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1}\right)$$

$$= \lim_{x \to a} x^{n-1} + \lim_{x \to a} x^{n-2}a + \ldots + \lim_{x \to a} xa^{n-2} + \lim_{x \to a} a^{n-1} = a^{n-1} + a^{n-1} + \ldots \text{n times } = na^{n-1}$$

Continuity of functions

- Let us now consider the closely related concept of continuity of functions. A function is continuous if the graph of the function has no breaks: within its domain, it is a continuous curve.
- A function y = f(x) is **continuous at a point** x = a in its domain if

$$\lim_{x \to a} f(x) = f(a)$$

- So by definition, the limit of a continuous function as x approaches "a" is the same as f(a), a fact that we have used before to evaluate limits.
  - Note that continuity (unlike limits) can be defined for a point.
- At an endpoint of the domain, the relevant one sided limit is used in the definition.
  - So at the left endpoint, it is the right hand limit; and at the right endpoint, it is the left hand limit. This means that if the function domain is [a, b], then at x = a and x = b, the function is continuous if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to b^-} f(x) = f(b)$$

- A function is **continuous on an interval** if it is continuous at every point in the interval.
- A **continuous function** f(x) is continuous at every point in its domain.
  - This does not imply that f(x) is continuous on any interval, because this may include points outside the domain. However f(x) is continuous on any interval fully contained in its domain.
- If a function f(x) is not continuous at x = a, we say it is **discontinuous** at x = a, and "a" is a **point of discontinuity** of f.

Continuity of functions (continued)

- So if x = a is a point of discontinuity for f(x), then one of the following holds:
  - limit of f(x) as x approaches “a” does not exist
  - the limit exists, but is either not equal to f(a), or f(a) is undefined.
    - When f(a) is undefined, the point x = a is not part of the function domain. But we can still define the limit of f(x) as x approaches “a”, if f(x) is defined near “a”.
- Some examples are given below:
  - Polynomial function P(x) is continuous, as well as the rational function P(x) / Q(x).
    - Note points where Q(x) = 0 are not part of the function domain. So f(x) = 1/x is a continuous function.
  - Absolute value function |x| is continuous.
  - The greatest integer function is discontinuous at all integer values.
  - The unit step function u(x) is discontinuous at x = 0 (everywhere else it is continuous).

Question: Determine if the following function is continuous

$f(x) = -2x + 3$ for $x < 0$ and
$\quad = (\sqrt{3} - x)^2$ for $x \geq 0$.

Solution: Both $(-2x + 3)$ and $(\sqrt{3} - x)^2$ are polynomials, hence they are continuous. But near x = 0, the different definitions of f(x) may not approach the same value, hence there can be a break. However this is not so (therefore the function is continuous), since

$$\lim_{x \to 0^-} (-2x+3) = 3 \text{ and } \lim_{x \to 0^+} \left(\sqrt{3} - x\right)^2 = 3 \text{ and } f(0) = 3.$$

Continuity rules

If functions f(x) and g(x) are continuous at x = a, then the following combinations are also continuous at x = a.

a) $(f+g)(x) = f(x)+g(x)$ b) $(f-g)(x) = f(x)-g(x)$

c) $(f \cdot g)(x) = f(x) \cdot g(x)$ d) $\left(\frac{f}{g}\right)$(x) or $\left(\frac{g}{f}\right)$(x) provided the denominator is not 0.

e) $f^{r/s}$ or $g^{r/s}$ where r and s are integers (assuming the definition makes sense).

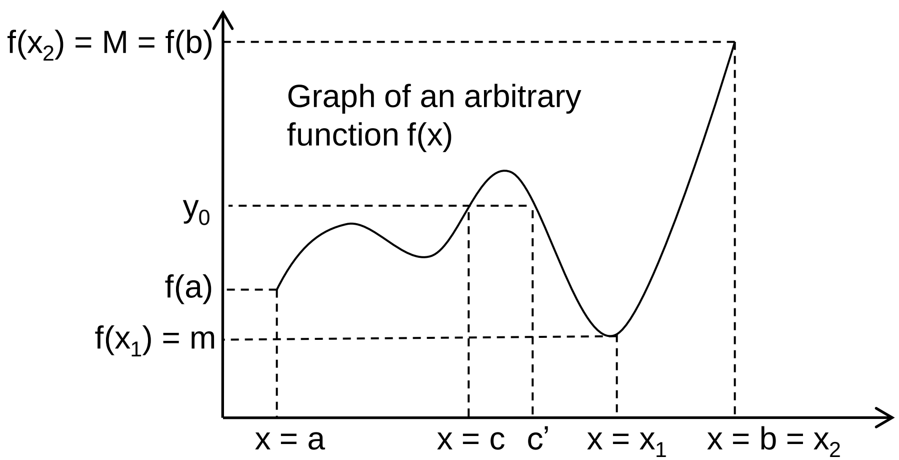
This readily follows from the limit rules, for example

$$
\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}\left(f(x)+g(x)\right) \text{ by definition of function sum} \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \text{ by limit rule for a sum of two functions} \\
&= f(a)+g(a) \text{ since f and g are continuous at } x=a \\
&= (f+g)(a) \text{ so the sum function is continuous at } x=a
\end{aligned}
$$

Function composition: If g(x) is continuous at x = a, and f(x) is continuous at g(a), then the composite f(g(x)) is continuous at x = a.

Properties of continuous functions

- **Intermediate Value Theorem**: If f(x) is continuous on a closed interval [a, b], then it takes on every value between f(a) and f(b).
  - This means that if y_0 is some value between f(a) and f(b), then there exists a point x = c in [a, b] such that $f(c) = y_0$.
- **Extreme Value Theorem**: If f(x) is continuous on a closed interval [a, b], then f(x) has an absolute minimum m and an absolute maximum M in [a, b]. This
  means that:
  - There exists x_1 and x_2 in [a, b] such that $f(x_1) = m$ and $f(x_2) = M$, and for all other x in [a, b], $m \leq f(x) \leq M$
  - With the previous theorem, it implies that f(x) takes on every value between m and M in [a, b].



Explanation on the two theorems

- The conditions stated in the Intermediate Value Theorem (IVT) and Extreme Value Theorem (EVT) are necessary, as we see below.
- Consider f(x) = x in [0, 1) and (x + 1) in [1, 2].
  - f(0) = 0 and f(2) = 3. But f(x) does not have any value in [1, 2) because of the discontinuity at x = 1. We cannot apply IVT to the interval [0 , 2], but we can apply it to the interval [1, 2].
- Consider f(x) = x defined on (0, 2). The interval is open, and the function doesn't have an absolute maximum or minimum value.
  - If the domain is changed to (0, 2], then f(x) has an absolute maximum of 2 at x = 2, but it has no absolute minimum.
  - If the domain is changed to [0, 2], then f(x) also has a minimum of 0 at x = 0.
    - Think about it, if the above is not clear. For example, in the open interval (0, 2), we can make x as "close to 0" as we like, but we still have an infinite number of points (like x/2, x/3,...) which are smaller than x; hence there is no minimum.
  - If f(x) = x in [0, 2) and f(2) = 0 producing a discontinuity at x = 2, then f(x) has no absolute maximum.
  - In summary, to apply EVT to an interval, the interval must be closed, and f(x) must be continuous on that interval.

Optional: What follows is a more formal look at the limit concept (needs prior knowledge of inequalities, involving absolute values).

Formal definition of limit

We said that a function f(x) has a limit L as x approaches "a", if f(x) can be made as close to L as we like, for all x sufficiently close to "a".

The formal definition given below defines "closeness" in a precise way.

The function f(x) has a limit L as x approaches "a", if given any positive ε (however small), we can find a positive δ, such that |f(x) – L| < ε when 0 < |x – a| < δ.

|f(x) – L| < ε means that f(x) lies in the open interval (L – ε, L + ε). Let us call this "interval 1". By making ε smaller, interval 1 becomes smaller, and f(x) stays closer to L.

Similarly 0 < |x – a| < δ means that x lies in the open interval (a – δ, a + δ) but is never equal to "a". Let us call this "interval 2". By making δ smaller, interval 2 becomes smaller, and x stays closer to "a".

By choosing ε, we fix interval 1 in which we want f(x) to be present. The definition says that when the limit exists, we can find an interval 2 around "a", such that for all values of x in interval 2; f(x) stays within interval 1. In other words, f(x) can be kept as close to L as we like (less than ε away from L), for all x sufficiently close to "a" (less than δ away from "a").

Note the definition does not help us to find L, but validates whether a given L is the limit (though the procedure is not always straightforward).

Example: Consider $\lim_{x \to 1}(2x+3) = 2 \times 1 + 3 = 5$. Let us use the limit definition to prove that 5 is the limit.

$$\left|f(x) - L\right| < \varepsilon \to \left|2x+3-5\right| = 2\left|x-1\right| < \varepsilon \to \left|x-1\right| < \varepsilon / 2$$

Note the last inequality helps us to define interval 2. If δ = ε/2 or smaller, then |f(x) – 5| < ε when 0 < |x – 1| < δ (which proves that 5 is the limit).

Formal definition (continued)

Example: Prove $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for $a > 0$.

Solution: We need to find δ such that $\left|\sqrt{x} - \sqrt{a}\right| < \varepsilon$ when $0 < |x - a| < \delta$.

$$\left|\sqrt{x} - \sqrt{a}\right| < \varepsilon \to \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon.$$

If we find a positive constant C, such that $\sqrt{x} + \sqrt{a} > C$, and if we choose $\delta = C\varepsilon$ or smaller,

we have $|x - a| < \delta \to |x - a| < C\varepsilon \to \dfrac{|x-a|}{C} < \varepsilon \to \dfrac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon \to \left|\sqrt{x} - \sqrt{a}\right| < \varepsilon$

Note the basic idea is to relate |f(x) – L| to |x – a|. Since √x > 0, we can set C = √a, and δ = ε√a or smaller (proving that √a is the limit).

The function √x is defined for all x ≥ 0, so at x = 0, only the right hand limit exists. A function f(x) has a right hand limit L as x approaches "a", if given any positive ε, we can find a positive δ, such that |f(x) – L| < ε when 0 < x – a < δ (or equivalently a < x < a + δ). Let us formally show that the right hand limit of √x is 0, as x approaches 0.

|√x – 0| < ε → x < ε^2. Therefore δ can be chosen as ε^2 or smaller; which means that the right hand limit is 0. The above discussion also shows that √x is a continuous function.

We can come up with similar formal definitions for other scenarios: For example, we say that f(x) approaches infinity as x approaches "a", if for any positive M (however large), we can find a positive δ, such that f(x) > M when 0 < |x – a| < δ.