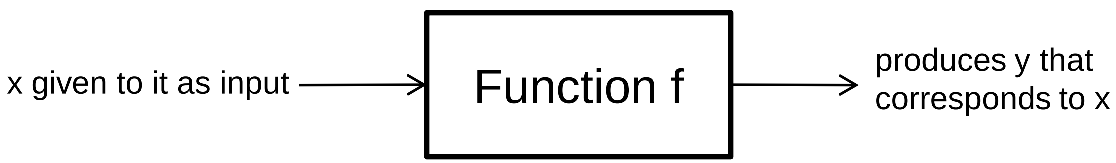
Lesson 03: Functions

Functions – an informal viewpoint

- Whenever one value “y” depends on (or is determined by) another value “x”, we say that y is a function of x. e.g.
  - Area A of a circle is a function of its radius r.
  - The temperature T at a place on a given day, is a function of time during the day (t).
  - The height (H) of a person is a function of the person’s age (x).
- The function can be considered as a rule or a formula that gives the value y, when the value x is given to it.

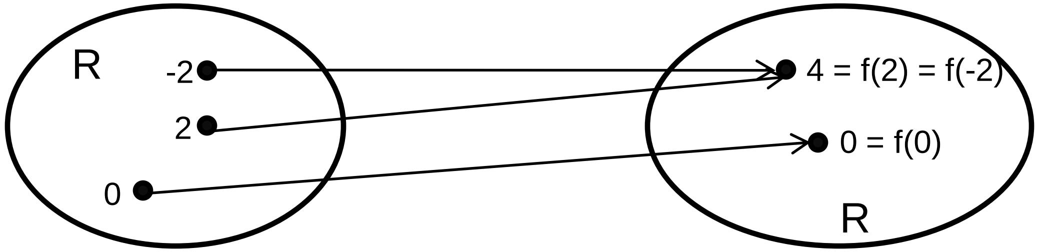


Function definition

- A **set** is a collection of objects. These objects are called **elements** of the set e.g.
  - A = {1, 4, 9, 16, 25, 36, 49} or equivalently A = {x: x= n^2, where n is a natural number and $1 \leq n \leq 7$}
  - B = {a, e, i, o, u}
  - C = {x: $0 \leq x < \infty$}
  - D = [0, 1]   (equivalent to {x: $0 \leq x \leq 1$})
  - E = (0, 2]   (equivalent to {x: $0 < x \leq 2$})
- A **function** f from set A to set B is a rule that assigns a *unique* (single) element f(x) in B to each element x in A. e.g.
  - $y = f(x) = x^2$
    - f is the symbol used to denote the function. f(x) read as "f of x" or the "value of f at x" is the value that f assigns to any x in set A. Since $f(x) = x^2$ here, the function f assigns the value x^2 to each x in A.
    - x and y are variables. Since y = f(x), y takes the value that the function f assigns to x in A. x can take on any value in A and the corresponding value of y depends on x (with the function f determining the value of y for a given x). Therefore, x is called an **independent variable** while y is called a **dependent variable**.

Function definition (continued)

- A function f from set A to set B is also denoted as f : A → B. e.g.
  - $f : R \rightarrow R$ and $y = f(x) = x^2$ (R denotes the set of all real numbers).
- A and B are usually sets of real numbers (like sets C, D and E above).
- Function is denoted by letters like f, g, h etc.
- Set A is called the **domain** of the function. When not stated explicitly, it is the largest set of real values x for which f(x) is real.
- All values that f(x) can have (as x varies through the domain) is called the **range** of the function. Range is a subset of B.
- For $y = f(x) = x^2$, the domain is all real values x $(-\infty, \infty)$, while the range is all real values ≥ 0 $[0, \infty)$.
- Different values of x can be assigned to the same value by f. e.g.
  - $f(x_1) = f(x_2)$ for $x_1 \neq x_2$ is a valid assignment by f.
- An **arrow diagram** has an arrow from x in set A to f(x) in set B to pictorially show the assignment done by the function. e.g. for $f(x) = x^2$

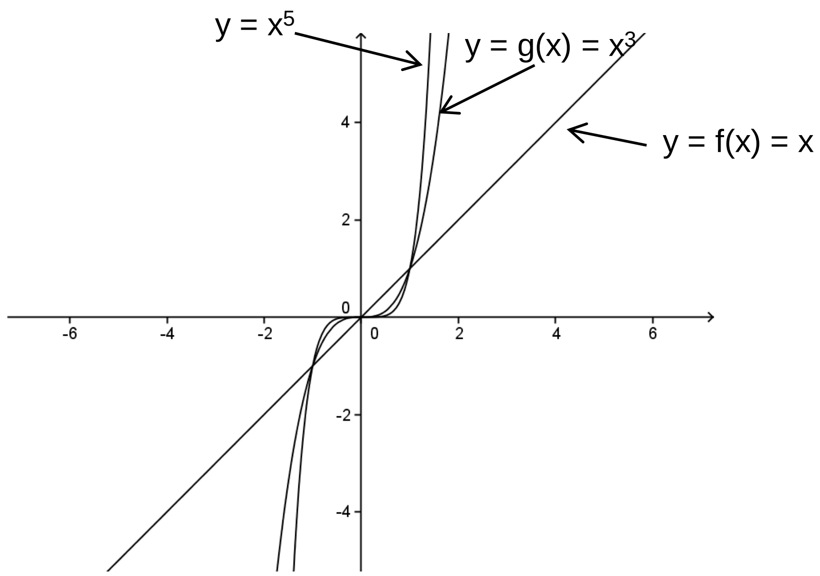


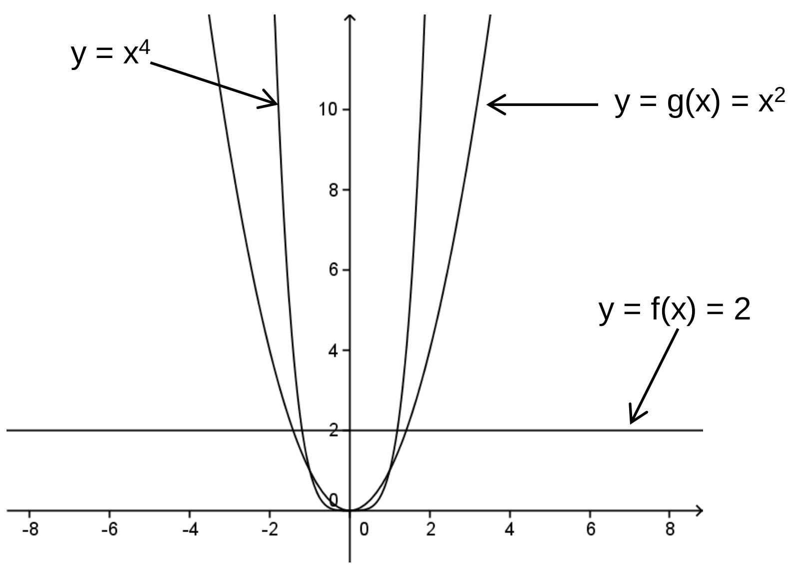
Function examples

- Area A of a circle is a function of its radius r
  - $A = f(r)$: This says that the variable A is the value of function f at r. But what is function f?
  - $f(r) = \pi r^2$: This says that the value of function f at r is equal to πr^2. Now the function is defined.
  - $A = f(r) = \pi r^2$: This says that the variable A is equal to the value of the function f at r, which is equal to πr^2. Thus, we have expressed the circle area A as a function of radius r.
- Consider $f(x) = 3x^2 + 8$
  - $f(1) = 3.(1)^2 + 8 = 11$
  - $f(2) = 3.(2)^2 + 8 = 20$
  - Domain of the function is all real values x.
  - Range is $[8, \infty)$. Why? $x^2 \geq 0$, so the min value of $3x^2 + 8$ is 8.

Some functions and their graphs

Graph of a function f(x) is the set of all points (x, y) in the coordinate plane such that y = f(x). x takes all values in the domain of f(x).





Odd Powers of x
Note range for these functions is (-∞, ∞)

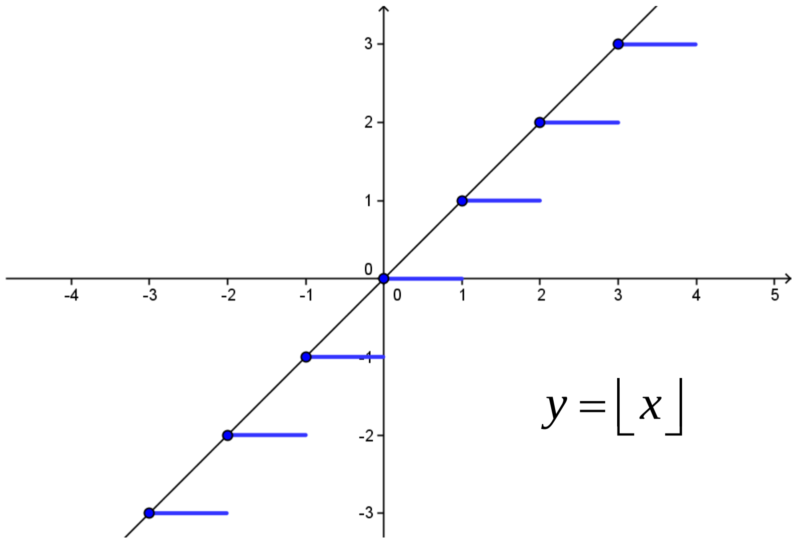
Even Powers of x
Note range for these functions (except the constant function) is [0, ∞)

Polynomial function of degree n has the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ ($a_n \neq 0$)

Common functions (continued)

Greatest integer function written as $y = f(x) = \lfloor x \rfloor$ assigns to each x, the greatest integer less than or equal to x.

Note the range consists of all integers. Also y jumps from one integer to the next, e.g. from 2 to 3, as x changes from "a little less than 3" to 3.



The graph consists of the blue lines and the dots.

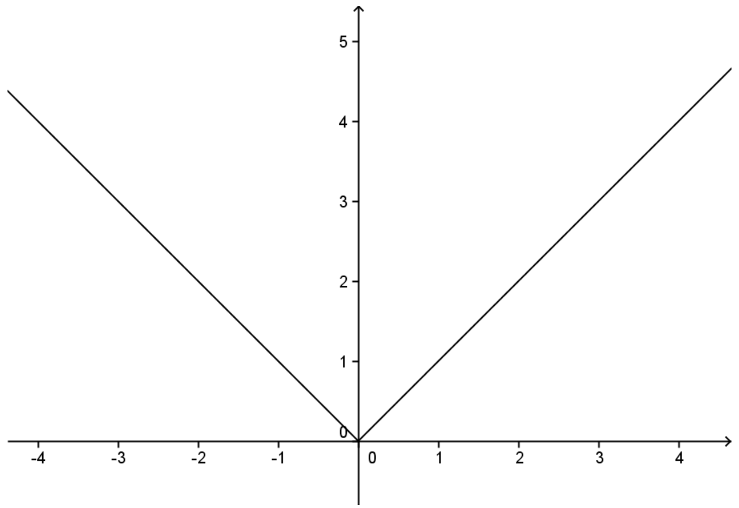
The sloping black line $y = x$, acts as a reference.

Dots (found at integer values) indicate that the point belongs the graph of $\lfloor x \rfloor$.

Absolute value function $y = f(x) = |x|$ is defined as

$y = x$, when x $\geq$ 0

$= -x$ when x < 0



y = |x|

Sum, difference, product and quotient of functions

For every x in the domain of both functions f(x) and g(x); the sum, difference, product and quotient are defined.

Sum $(f+g)(x) = f(x) + g(x)$

e.g. the value of the sum function (f + g) at x, is the value of the function f at x, plus the value of the function g at x.

Difference $(f-g)(x) = f(x) - g(x)$

Product $(fg)(x) = f(x)g(x)$

Quotient $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ Also, g(x) must not be 0

Example:

Let f(x) = $\sqrt{x+1}$ and g(x) = $\sqrt{4-x^2}$

Domain of f(x) is [-1, ∞) (since x + 1 ≥ 0)

Domain of g(x) is [-2, 2] (since 4 - $x^2 \geq 0$)

Points common to both domains [-1, 2]

Therefore,

$(f+g)(x) = \sqrt{x+1} + \sqrt{4-x^2}$ with domain [-1, 2]

$(f-g)(x) = \sqrt{x+1} - \sqrt{4-x^2}$ with domain [-1, 2]

$(fg)(x) = \sqrt{(x+1)(4-x^2)}$ with domain [-1, 2]

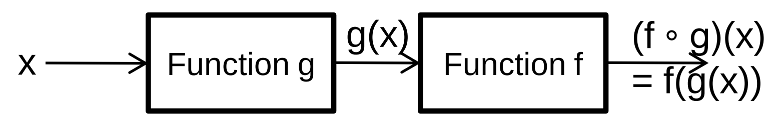
$\left(\dfrac{f}{g}\right)(x) = \sqrt{\dfrac{x+1}{4-x^2}}$ with domain [-1, 2) since 4 - x^2 must not be 0.

$\left(\dfrac{g}{f}\right)(x) = \sqrt{\dfrac{4-x^2}{x+1}}$ with domain (-1, 2] since $x+1$ must not be 0

Composite functions

If f(x) and g(x) are functions, then the composite function f ∘ g ("f composed with g") is defined as (f ∘ g)(x) = f(g(x)) [the value of function f at g(x)]

- Given a x in the domain of g
  - function g assigns (maps) it to g(x).
  - If g(x) lies in the domain of f, then f(x) assigns the value f(g(x)) to it.
- Therefore, composite (f ∘ g) is defined for all x where g(x) lies in the domain of f.
- (g ∘ f)(x) = g(f(x)) ≠ (f ∘ g)(x) = f(g(x))



Example

Let $f(x) = \sqrt{x+3}$ and $g(x) = 2x$

$(f \circ g)(x) = f(g(x)) = f(2x) = \sqrt{2x+3}$ with domain [-3/2, ∞)

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+3}) = 2\sqrt{x+3}$ with domain [-3, ∞)

$(f \circ f)(x) = f(f(x)) = f(\sqrt{x+3}) = \sqrt{\sqrt{x+3}+3}$

Since $\sqrt{x+3} \geq 0$ for x ∈ [-3, ∞), the domain is [-3, ∞)

$(g \circ g)(x) = g(g(x)) = g(2x) = 4x$ with domain (-∞,∞)