Lesson 12: Derivatives (Part 2)

Parametric curves

- Instead of representing a curve using y = f(x), both x and y coordinates can be a function of a third variable t. e.g. x = f(t) and y = g(t).
- t is called the **parameter** for the curve, and **parameter interval** refers to the set of allowed values of t.
- The set of points (x, y) = (f(t), g(t)) is called the **parametric curve**, and the equations of x and y (in terms of t) are called **parametric equations** of the curve.
  - Parametric representation is commonly used in physics, where the x, y (and z) coordinates of a particle are expressed as a function of time t.
  - In this course also, we have used parametric equations for conic sections, one of which is given below.

Example: $x = a\cos t \quad y = a\sin t$ where $t \in [0,\ 2\pi]$

represents a circle with center at origin and radius a because

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2$$

As t increases from 0 to 2π, we trace the circle counter-clockwise once starting at point $(a,\ 0)$ and ending again at it.

Slope of a parametric curve

Let $x = f(t)$ and $y = g(t)$. Taking y to be an (implicit) function of x, where x is a function of t, we can apply the chain rule to write

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Solving for dy/dx, we have $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

Since $\dfrac{d^2 y}{dx^2} = \dfrac{dy'}{dx} = \dfrac{dy'}{dt} \Big/ \dfrac{dx}{dt}$ where $y' = \dfrac{dy}{dx}$

Example: Let $x = 1 - 8t$ and $y = 6t + 5$ where $-\infty < t < \infty$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{6}{-8} = -\frac{3}{4}$$

Note this is just a line, as can be readily seen by eliminating t

$\dfrac{1-x}{8} = \dfrac{y-5}{6} \rightarrow y = -\dfrac{3}{4}x + \dfrac{23}{4}$ (slope matches the dy/dx value above).

As we will see in a later lesson, the standard form for the parametric equations of a line is $x = h + r\cos\theta$ and $y = k + r\sin\theta$ (where r can have all real values, and θ is the angle of inclination).

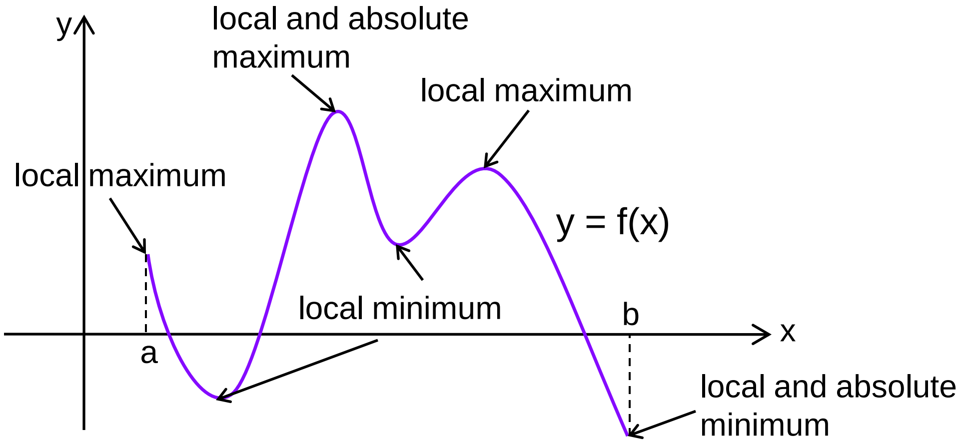
However by comparing with the standard form, we see that $r\cos\theta = -8t$ and $r\sin\theta = 6t$. Squaring both and adding gives $r^2 = 100t^2 \rightarrow t = r/10$, and we can write the equations as $x = 1 - (4/5)r$ and $y = 5 + (3/5)r$.

Extreme values of functions

- We will now consider an important application of derivatives; namely finding the minimum and maximum value of functions. To start with, we need some definitions.
- We know that if f(x) is continuous in [a, b], then there exist x_1 and x_2 in [a, b] such that $f(x_1) = m$ and $f(x_2) = M$, and for all other x in [a, b], $m \leq f(x) \leq M$ (Extreme Value Theorem seen in an earlier lesson).
- We call M as the **absolute maximum**, and m as the **absolute minimum** of f(x) in [a, b]. Absolute maximum and absolute minimum values are also called **global** (or **absolute**) **extrema** values.
- A function has a **local maximum** at an interior point c of its domain, if $f(x) \leq f(c)$ for all x in some open interval containing c. Similarly, a function has a **local minimum** at an interior point c of its domain, if $f(x) \geq f(c)$ for all x in some open interval containing c.
- Local minimum and local maximum values are also called **local** (or **relative**) **extrema** (plural of extremum) values.

At end points “a” and b, we use a half open interval to define local extrema. e.g. x = a has a local minimum, if $f(a) \leq f(x)$ for all x in some half open interval [a, d).

Note every absolute extremum point is also a local extremum point.



First derivative theorem for local extrema

- If f(x) has a local extremum (maximum or minimum) at an interior point c of its domain and f'(c) exists, then f'(c) = 0.
  - Seems reasonable from the sample graph, since the maximum occurs at a peak, and the minimum at a valley (trough), and the tangent appears horizontal at these points.
- Its proof follows from derivative and local extrema definitions. So let f(x) have a local minimum at x = c (the proof for local maximum is similar). We then have,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

For a minimum point $f(c) \le f(x)$ for x close to c, hence $f(x) - f(c) \ge 0$

and $x - c > 0$ for $x > c$ and $x - c < 0$ for $x < c$

$$\therefore \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \ge 0 \quad \text{and} \quad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \le 0$$

Since the derivative exists at c, both right and left hand limits must be equal, which implies that both must be zero, i.e. $f'(c) = 0$.

First derivative theorem for local extrema (continued)

- So the only points in the domain of f(x), where extrema (local or global) can occur are:
  - Interior points where the derivative is 0.
  - Interior points where derivative doesn’t exist.
  - End points of the domain.
- Interior points of f(x) where the derivative is zero or does not exist are called **critical points** of f(x).
- To find the absolute maximum and minimum of a function continuous on a finite closed interval, is now straight-forward:
  - Evaluate f(x) at all the points mentioned above.
  - The point where f(x) has the minimum value is the absolute minimum; where it has the maximum value is the absolute maximum (note we still don’t have a rule to say, if the other critical points or endpoints have a local maximum or minimum).
  - The conditions “continuous and finite closed interval” are those of the Extreme Value Theorem.

Rolle's theorem and Mean Value Theorem

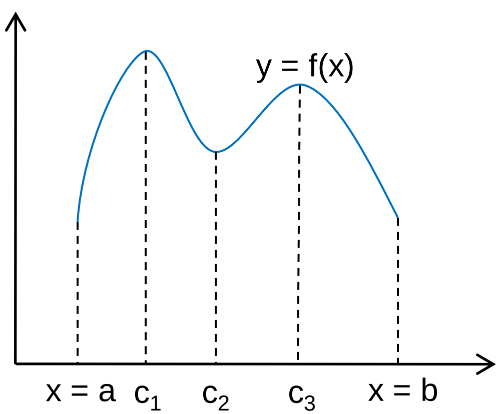
Rolle's Theorem: If f(x) is continuous in [a, b] and differentiable in (a, b) and f(a) = f(b), then there is <u>at least one</u> point c in (a, b), where f'(c) = 0.

Means that when f(a) = f(b), and the graph is a smooth curve between x = a and x = b, there is at least one point in between where the tangent is horizontal.

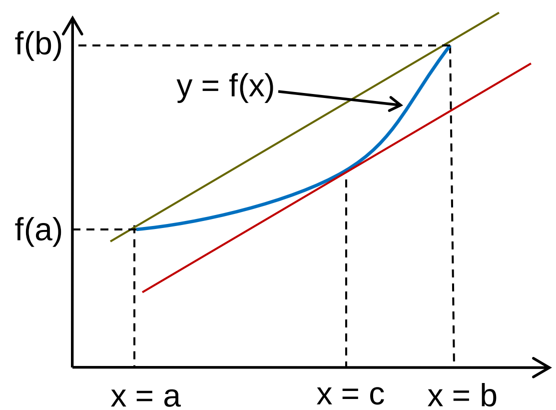
Mean Value Theorem: If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there is at least one point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Note [f(b) – f(a)] / (b – a) is the slope of the secant through the points (a, f(a)) and (b, f(b)) on the graph. So if the graph between these points is a smooth curve, then there is some point in between, where the tangent is parallel to this secant.

Both the theorems seem intuitively reasonable, and we will skip the formal proof.



Points c of Rolle's theorem



Point c of Mean Value theorem

Consequences of the mean value theorem

- Corollary 1: If f'(x) = 0 in an interval (a, b) then f(x) = c in that interval, where c is a constant.
  - The converse is obvious; if f(x) = c, then f'(x) = 0.
- Corollary 2: If f'(x) = g'(x) on an interval (a, b), then f(x) = g(x) + C on that interval, where C is a constant.
  - Once again the converse is obvious; if f(x) = g(x) + C, then f'(x) = g'(x).

Proof of Corollary 1

Take any two points x_1 and x_2 in $(a,\ b)$ where $x_1 < x_2$. Since $f'(x) = 0$ on $(a,\ b)$,

$f(x)$ satisfies the conditions of the mean value theorem on $[x_1,\ x_2]$.

$\therefore\ f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c in $(x_1,\ x_2)$.

$f'(c) = 0$ since $f'(x) = 0$ throughout $(a,\ b) \rightarrow f(x_2) = f(x_1)$

Similarly x_1 is equal to every other point x_i in $(a,\ b)$ thus proving that $f(x)$ is constant in $(a,\ b)$.

Proof of Corollary 2

Let $h(x) = f(x) - g(x)$. Then at all points in $(a,\ b)$

$h'(x) = f'(x) - g'(x) = 0$ since $f'(x) = g'(x)$ on $(a,\ b)$

By Corollary 1, $h(x) = C$ (a constant) $\rightarrow\ f(x) = g(x) + C$

Mean value theorem consequences: increasing and decreasing functions

- f(x) is said to be **increasing** on an interval [a, b] if f(x) increases in value as x increases, i.e. for any pair of points x_1 and x_2, $f(x_1) < f(x_2)$ when $x_1 < x_2$.
  - Definition implies that Δx and Δy have the same sign, hence $f'(x) > 0$ on (a, b).
- f(x) is said to be **decreasing** on an interval [a, b] if f(x) decreases in value as x increases, i.e. for any pair of points x_1 and x_2, $f(x_1) > f(x_2)$ when $x_1 < x_2$.
  - Definition implies that Δx and Δy have opposite signs, hence $f'(x) < 0$ on (a, b).
- A function that is either increasing or decreasing (but not both) on an interval is said to be **monotonic** on that interval.
- Corollary 3: Let f(x) be continuous on [a, b] and differentiable on (a, b).
  - If $f'(x) > 0$ on (a, b), then f(x) is increasing on [a, b].
  - If $f'(x) < 0$ on (a, b), then f(x) is decreasing on [a, b].

Proof of Corollary 3

Let x_1 and x_2 be two points in $[a, b]$ with $x_1 < x_2$. Applying the mean value theorem to $[x_1, x_2]$

$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c in (x_1, x_2).

Since $x_2 - x_1 > 0$, the right side has the same sign as $f'(c)$.

If $f'(x) > 0$ on (a, b), then $f'(c) > 0 \rightarrow f(x_2) > f(x_1)$ that is function is increasing.

If $f'(x) < 0$ on (a, b), then $f'(c) < 0 \rightarrow f(x_2) < f(x_1)$ that is function is decreasing.

Increasing and decreasing functions (continued)

- The definition for increasing and decreasing functions given by us is sometimes called *strictly increasing* and *strictly decreasing* respectively (NCERT text as well as IIT questions use this terminology).
- In the context of this terminology:
  - $f(x)$ is increasing if $f(x_1) \le f(x_2)$ for $x_1 < x_2$. Note the use of "less than or equal to". So $f'(x) \ge 0$ for an increasing function (while $f'(x) > 0$ for a strictly increasing function).
  - $f(x)$ is decreasing if $f(x_1) \ge f(x_2)$ for $x_1 < x_2$. So $f'(x) \le 0$ for a decreasing function.
- When a function $f(x)$ is monotonic (in the strict sense), then $f(x)$ is one to one. Recall that a function is one to one, when each x in the domain is mapped to a different value by f (so if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$).

First derivative test for distinguishing local extrema

- Let c be a critical point of a function f(x), which is continuous in some interval I around c, and f'(x) exists in I, except possibly at c. As x increases from values less than c, to values more than c in I,
  - If f' changes from positive to negative, then f has a local maximum at c.
    - Implies f(x) is increasing to the left of c, and decreasing to the right of c, so we expect c to be a local maximum (see Fig 1).
  - If f' changes from negative to positive, then f has a local minimum at c.
    - Implies f(x) is decreasing to the left of c and increasing to the right of c, so we expect c to be a local minimum (Fig 2).
  - If f' has the same sign on either side of c, then f has no local extremum at c.
    - f(x) is increasing or decreasing on both sides of c, hence there is no extremum at c (Fig 3).

x = 0 is a critical point where f' = 0 (a maximum point)

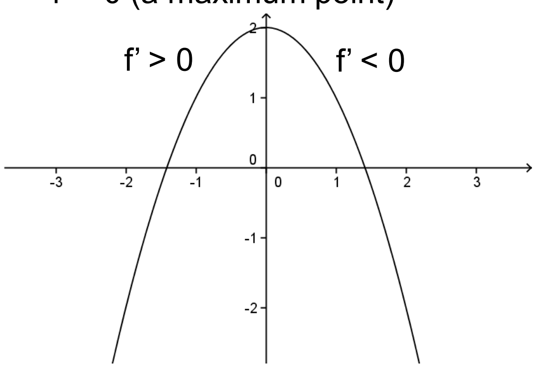
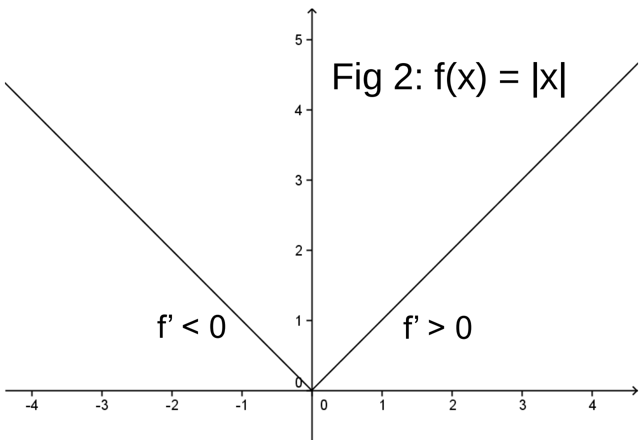
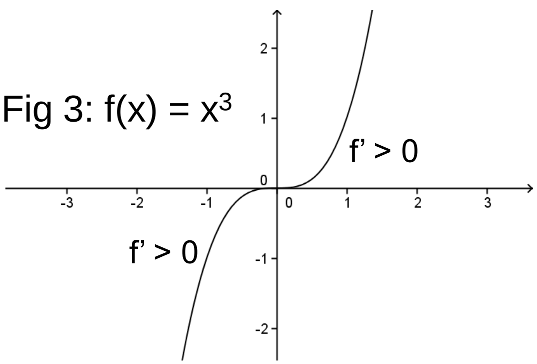


Fig 1: $f(x) = 2 - x^2$



x = 0 is a critical point where f' does not exist (a minimum point)

Fig 3: $f(x) = x^3$



x = 0 is a critical point where f'(x) = 0 but is not a local extremum

Second derivative test for distinguishing local extrema

- If f'(c) = 0 and f''(x) is continuous in an open interval that contains x = c,
  - If f''(c) < 0, then f(x) has a local maximum at x = c
  - If f''(c) > 0, then f(x) has a local minimum at x = c
  - If f''(c) = 0, then the test fails – the point may have a local minimum or maximum or neither.

- Proof
  - Consider f''(c) < 0 scenario. Since f'' is continuous around c, there is some interval I around c where f''(x) < 0. This implies f'(x) is decreasing in I. Since f'(c) = 0, f'(x) > 0 for x < c in I and f'(x) < 0 for x > c in I. So f'(x) changes from positive to negative around c, and by the first derivative test, there is a local maximum at x = c.
  - The proof for f''(c) > 0 is similar.
  - When f''(c) = 0, we can't conclude anything. e.g. at x = 0, f(x) = x^3 has no extremum, whereas f(x) = x^4 has a minimum. For this reason, we usually use the first derivative test in our examples.

Concavity and point of inflection

- A function f(x) is called **concave up** on an open interval I, if f'(x) is increasing in I. This means that f''(x) > 0 in I. Also the tangent to f(x) at any point P in I, is below f(x) around P (see Fig 2).
- Similarly a function f(x) is called **concave down** on an open interval I, if f'(x) is decreasing in I. This means that f''(x) < 0 in I. The tangent to f(x) at any point Q in I, is above f(x) around Q.
- "Concave up" and "concave down" refers to the shape of graph in I. For example,
  - If $f(x) = x^2$, then f''(x) = 2. So the graph is concave up for all x, and this seems reasonable from its graph.
  - If $f(x) = x^3$, then f''(x) = 6x. So the graph is concave up for x > 0, and concave down for x < 0. The point x = 0, around which the concavity of the curve changes, is called a **point of inflection**. At the point of inflection, f''(x) = 0 or it does not exist.
  - The change in the sign of f''(x) around the inflection point, means that the point is a local extremum for f'(x) (by the first derivative test applied to f'(x)).
- At a local maximum (assuming f''(x) exists), we expect f(x) to be concave down, so f''(x) < 0 (consistent with the second derivative test). Similarly at a local minimum, f(x) must be concave up, hence f''(x) > 0.

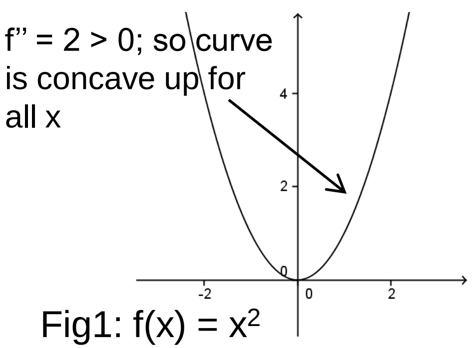


Fig1: $f(x) = x^2$

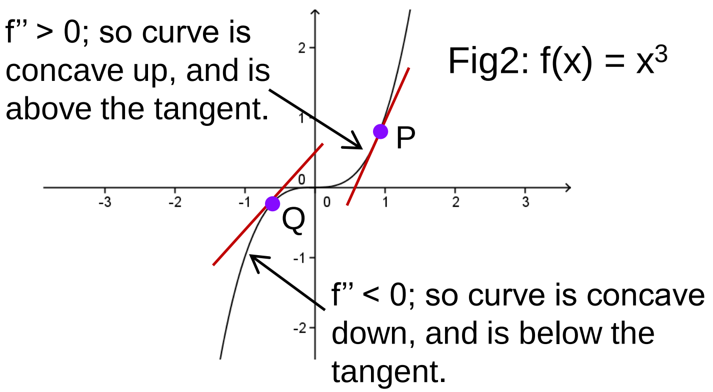


Fig2: $f(x) = x^3$

Examples

Example 1 [IIT, 2008]: The total number of local maxima and minima of the function

$f(x) = (2+x)^3, \quad -3 < x \le -1$

$= x^{2/3}, \qquad -1 < x < 2$

is A) 0 B) 1 C) 2 D) 3

Solution: Let $u = (2+x)^3$ for $-3 < x \le -1$

$\frac{du}{dx} = 3(2+x)^2 = 0$ when $x = -2$

But $\frac{du}{dx}$ is always non-negative and does not change sign at $x = -2$, hence this is not a local extremum.

Let $v = x^{2/3}$ for $-1 < x < 2$

$\frac{dv}{dx} = \frac{2}{3}x^{-1/3}$ so it is undefined at $x = 0$

Also $\frac{dv}{dx}$ is negative for $x < 0$ and positive for $x > 0$, hence $x = 0$ is local minimum.

Given the different definitions of $f(x)$ on either side of $x = -1$, we need to test this point as well. You can verify that $f(x)$ is continuous here, and from a differentiability viewpoint, we have

$\lim_{x \to -1^-} \frac{du}{dx} = 3 \qquad \lim_{x \to -1^+} \frac{dv}{dx} = -\frac{2}{3}$

Hence $f(x)$ is not differentiable at $x = -1$ but given that it increases before -1 and decreases after -1, $x = -1$ is a local maximum. Therefore the answer is C) 2.

Examples (continued)

Some more explanation is required for Example 1. A maximum occurs at $x = -1$, because it is continuous there, and the derivative changes sign.

Consider $f(x) = x$ in $[0, 1]$ and $(-x + 3)$ in $(1, 3]$.

The derivative changes from 1 to -1, as we go through $x = 1$. But $x = 1$ is <u>not</u> a local maximum, because $f(1) < f(x)$ for x to the immediate right of $x = 1$, because of the discontinuity at that point.

At $x = 1$, is the left hand derivative (LHD) equal to 1, and the right hand derivative (RHD) equal to -1, given that the two functions on either side of the point, have this slope?

LHD is correct, but RHD is wrong. If you use the definition of RHD, you will see that RHD is undefined at $x = 1$.

However in example 1), we evaluated the LHD and RHD at $x = -1$, using the limiting value of the derivative of $u(x)$ and $v(x)$ respectively (the functions defined on either side). This gives the correct answer, because $f(x)$ is continuous at $x = -1$ (unlike the above example).

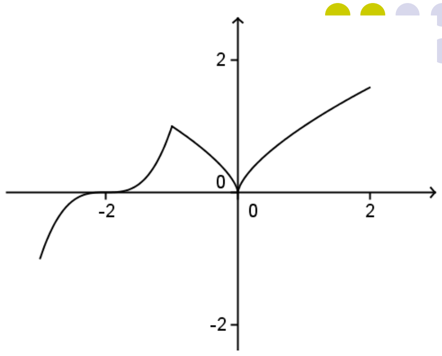
Usually for problems involving simple functions, it is quicker and easier to solve them using graphs. If you have taken our course, this should be pretty straight-forward, but an explanation follows on next page.

Examples (continued)

Example 1 (graphical solution): $(2 + x)^3$ is x^3 shifted to the left by 2 units. The function always increases, so we expect no local extremum from −3 to −1.

$x^{2/3}$ is the third root of x^2, so it is symmetric about the y-axis. We expect it to decrease from −1 to 0, and then increase from 0 to 2; so a minimum is expected at x = 0. From an extremum viewpoint, the concavity doesn't matter.

Near the boundary point x = −1, both functions approach the value 1. $(2 + x)^3$ is increasing on the left, while $x^{2/3}$ is decreasing on the right, so a maximum is expected at x = −1.
Hence there are two local maxima and minima.



Graph of f(x)

Example 2 [IIT 2007] In the following [x] represents the great integer less than or equal to x. Match the functions in Column 1 with their properties in Column 2.

Column 1	Column 2
A) $x\|x\|$	p) continuous in (-1, 1)
B) $\|x\|^{1/2}$	q) differentiable in (-1, 1)
C) x + [x]	r) strictly increasing in (-1, 1)
D) $\|x – 1\| + \|x + 1\|$	s) not differentiable at least at one point in (-1, 1)

Solution: Note we need to analyze each function in (−1, 1) for continuity, differentiability and increasing behavior. We will analyze one function at a time. Continued on next page...

Examples (continued)

A) $f(x) = x|x| = x^2$ for $x \geq 0$ and $-x^2$ for $x < 0$.

x^2 and $-x^2$ are continuous, and both approach $f(0) = 0$ as $x \to 0$. So $f(x)$ is continuous.

$f'(x) = 2x$ for $x \geq 0$ and $-2x$ for $x < 0$. Both approach 0 as $x \to 0$. Hence $f(x)$ is differentiable.

For checking differentiability at x = 0, we can also start with the definition of Lf'(0) and Rf'(0), the left and right hand derivative respectively at x = 0.

$$Lf'(0) = \lim_{h\to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0-} \frac{-h^2 - 0}{h} = \lim_{h\to 0-} -h = 0 \quad \text{and} \quad Rf'(0) = \lim_{h\to 0+} \frac{h^2 - 0}{h} = \lim_{h\to 0+} h = 0.$$

Since RHD and LHD are both zero; f(x) is differentiable at x = 0, apart from other points in (−1, 1).

f'(x) is always positive, hence f(x) is strictly increasing (at x = 0, f'(x) = 0, but f(x) = 0 only here).
Therefore properties p, q and r hold for function A.

B) $f(x) = \sqrt{|x|} = x^{1/2}$ for $x \geq 0$ and $(-x)^{1/2}$ for $x < 0$

Both $x^{1/2}$ and $(-x)^{1/2}$ are continuous, approaching $f(0) = 0$ as $x \to 0$. So $f(x)$ is continuous.

$f'(x) = \frac{1}{2}x^{-1/2}$ for $x > 0$ and $-\frac{1}{2}(-x)^{-1/2}$ for $x < 0$. Since $f'(x) \to \infty$ as $x \to 0^+$ and $-\infty$ as $x \to 0^-$,

$f(x)$ is not differentiable at $x = 0$. It is differentiable elsewhere in $(-1, 1)$.

Also $f'(x) > 0$ for $x > 0$ and it is < 0 for $x < 0$, $f(x)$ decreases in $(-1, 0)$ and increases in $(0, 1)$.

Therefore, properties p and s are true for function B.

Examples (continued)

As mentioned earlier, a rough graph of f(x) will help us to find the properties quickly. e.g. f(x) in B) is just $\sqrt{x}$ for $x \ge 0$, and a reflection of it in the y-axis for $x < 0$.

C) $f(x) = x + [x] = x - 1$ for $(-1, 0)$ and x for $[0, 1)$

$f(x) \to -1$ as $x \to 0^-$ and 0 as $x \to 0^+$. Hence function is not continuous $x = 0$. Therefore, it is also not differentiable at $x = 0$. Since $f'(x) = 1$ elsewhere, function is increasing in $(-1, 1)$.

Therefore, properties r and s are true for function C.

D) $f(x) = |x-1| + |x+1| = 1 - x + x + 1 = 2$ in $(-1, 1)$.

Hence it is differentiable and continuous in $(-1, 1)$. Being a constant, it is not increasing.

Therefore, properties p and q apply to function D.

Example 3 [IIT 2005]: Let f(x) be a twice differentiable function such that $f(x) = x^2$ for x = 1, 2, 3; then

a) f''(x) = 2 for all x in (1, 3) b) f''(x) = 2 for some x in (1, 3)
c) f''(x) = 3 for all x in (2, 3) d) f''(x) = f'(x) for some x in (2, 3)

Solution: Consider the function $g(x) = f(x) - x^2$. Note that g(1) = g(2) = g(3) = 0; and g(x) is twice differentiable.

So we can apply Rolle's theorem to g(x) in [1, 2] to say that $g'(c_1) = 0$ for some c_1 in (1, 2). We can also apply the theorem to [2, 3] to say that $g'(c_2) = 0$ for some c_2 in (2, 3).

Next we can apply Rolle's theorem to g'(x) in $[c_1, c_2]$ to say that that $g''(c_3) = 0$ for some c_3 in (c_1, c_2). Since (c_1, c_2) is contained in (1, 3), we can also say that c_3 belongs to (1, 3).

But g''(x) = f''(x) – 2; so $g''(c_3) = 0$ implies $f''(c_3) = 2$. So option b) is correct.

Examples (continued)

Example 4: A closed container is made from a cylinder of radius r and height h with a hemispherical dome at the top. What is the relationship between r and h that maximizes the volume for a given surface area?

Surface area $A = 2\pi rh + \pi r^2 + 2\pi r^2$ (cylinder side + base + top). So $h = \dfrac{A - 3\pi r^2}{2\pi r}$

Volume $V = \pi r^2 h + \dfrac{2}{3}\pi r^3$ (cylinder + top hemisphere)

$$V = \pi r^2 \left(\frac{A - 3\pi r^2}{2\pi r} \right) + \frac{2}{3}\pi r^3 = \frac{r}{2}(A - 3\pi r^2) + \frac{2}{3}\pi r^3 = \frac{Ar}{2} - \frac{5\pi r^3}{6}$$

$$\frac{dV}{dr} = \frac{A}{2} - \frac{5\pi r^2}{2} = 0 \rightarrow r_0 = \sqrt{\frac{A}{5\pi}}$$

Note for r < r_0, $\dfrac{dV}{dr} > 0$ and for r > r_0, $\dfrac{dV}{dr} < 0$, hence r_0 is a maximum.

$$h = \frac{A - 3\pi r^2}{2\pi r} = \frac{A - 3\pi \frac{A}{5\pi}}{2\pi \sqrt{\frac{A}{5\pi}}} = \frac{\frac{2A}{5}}{2\sqrt{\pi}\sqrt{\frac{A}{5}}} = \sqrt{\frac{A}{5\pi}} = r_0$$